

# Pure spinors, intrinsic torsion and curvature in even dimensions

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## Abstract

We develop a spinor calculus on a  $2m$ -dimensional complex Riemannian manifold  $(\mathcal{M}, g)$  equipped with a preferred projective pure spinor field. Such a projective spinor,  $[\xi]$  say, defines a distribution  $\mathcal{N}$  of totally null  $m$ -planes on  $\mathcal{M}$ , and the structure group of the frame bundle of  $(\mathcal{M}, g)$  is reduced to  $P$ , the Lie parabolic subgroup of  $\mathrm{SO}(2m, \mathbb{C})$  stabilising  $[\xi]$ . This leads to a higher-dimensional notion of a principal spinor, and on this basis, we give an algebraic classification of curvature tensors of the Levi-Civita connection of  $g$ , which, for the Weyl tensor, generalises the Petrov-Penrose classification of the (anti-)self-dual Weyl tensor from four to higher even dimensions.

In analogy to the Gray-Hervella classification of almost Hermitian manifolds, we classify the intrinsic torsion of the  $P$ -structure of the frame bundle in terms of irreducibles, thereby measuring the failure of the Levi-Civita connection to preserve  $[\xi]$ . The classification thus obtained encodes the geometric properties of  $\mathcal{N}$ , and also constitutes a complex Riemannian analogue of the notion of shearfree congruences of null geodesics in four-dimensional Lorentzian geometry.

We then study the relation between spinorial differential equations, such as the twistor equation, on pure spinor fields and the geometric properties of their associated null distributions. In particular, we give necessary and sufficient conditions for the null distribution of a pure twistor-spinor to be integrable. We finally conjecture a refined version of the complex Goldberg-Sachs theorem in higher dimensions.

Much of this work can be applied to the study of real smooth pseudo-Riemannian manifolds of signature  $(m, m)$  equipped with a preferred projective real pure spinor field.

## 1 Introduction and motivation

Petrov's classification of the Weyl tensor in four-dimensional general relativity [Pet00] has been an essential tool in discovering solutions to Einstein's field equations, and its usefulness is particularly well exemplified by the Goldberg-Sachs theorem [GS09] and its generalisations [KT62, RS63], which relate the algebraic speciality of the Weyl tensor to the existence of shearfree congruences of null geodesics on spacetimes, found notably in the Kerr black hole solution [Ker63].

It was however the spinorial approach to general relativity introduced by Witten [Wit59] and Penrose [Pen60] that not only simplified Petrov's classification in the computational sense, but also gave it a more tractable geometrical interpretation. In a nutshell (see [PR84, PR86] for details), this approach consists in classifying the multiplicities of the roots of the homogeneous quartic polynomial<sup>1</sup>

$$\Psi_{A'B'C'D'} \pi^{A'} \pi^{B'} \pi^{C'} \pi^{D'} = 0, \quad (1.1)$$

at every point of spacetime, where  $\Psi_{A'B'C'D'}$ , known as the (*self-dual*) *Weyl conformal spinor*, corresponds to the self-dual part of the Weyl tensor  $C_{abcd}$ . It is a complex, totally-symmetric, 4-spinor, and in Lorentzian signature, its complex conjugate is a symmetric spinor  $\tilde{\Psi}_{ABCD}$  of the opposite chirality, corresponding to the anti-self-dual part of  $C_{abcd}$ . A solution to (1.1) is known as a *principal spinor* of  $\Psi_{A'B'C'D'}$ . At any point of

<sup>1</sup>Here, we use the abstract index notation of [PR84]. Lower-case Roman indices refer to the four-dimensional standard representation of the special orthogonal group, and primed and unprimed upper case Roman indices to the two-complex-dimensional chiral positive and negative spinor representations respectively.

spacetime, the self-dual Weyl tensor generically admits four distinct principal spinors, and is known to be of Petrov-Penrose type  $\{1, 1, 1, 1\}$  (or type I). The algebraic speciality of the Weyl tensor is then characterised by the existence of a *repeated* principal spinor,  $\xi^{A'}$ , say, of  $\Psi_{A'B'C'D'}$ , or equivalently, a repeated root of (1.1), i.e. the Weyl conformal spinor satisfies what is known as the Petrov-Penrose type  $\{2, 1, 1\}$  (or type II) condition

$$\Psi_{A'B'C'D'}\xi^{A'}\xi^{B'}\xi^{C'} = 0. \quad (1.2)$$

In Lorentzian signature, the spinor  $\xi^{A'}$  and its complex conjugate  $\bar{\xi}^A$  give rise to a null vector  $k^a := \xi^{A'}\bar{\xi}^A$ , and any null vector arises in this way up to phase. The principal spinor  $\xi^{A'}$  is repeated if and only if  $k^a$  is a repeated principal null direction of  $C_{abcd}$ , i.e.  $k^ak^cC_{abc[d}k_{e]} = 0$ . If the spacetime is Einstein, then by the Goldberg-Sachs theorem,  $k^a$  generates a shearfree congruence of null geodesics. Equivalently, the spinor field  $\xi^{A'}$  locally satisfies

$$\xi^{B'}\xi^{A'}\nabla_{AA'}\xi_{B'} = 0, \quad (1.3)$$

where  $\nabla_{AB'}$  is the Levi-Civita connection. Geometrically, equation (1.3) means that  $\xi^{A'}$  is foliating, i.e. the field of totally null 2-planes annihilated by  $\xi^{A'}$  – these are spanned by vector fields of the form  $\xi^{A'}\alpha^A$  for some spinor field  $\alpha^A$  – is (locally) integrable. Equation (1.3) thus tells us that the projective spinor field  $\xi^{A'}$  is parallel along the null distribution it defines.

The Petrov-Penrose classification can also be applied to other metric signatures and complex Riemannian manifolds. The self-dual and anti-self-dual parts of the Weyl tensor will now be independent of each other, and the Petrov-Penrose classification will apply to either part separately. In any signature, the overarching theme remains the interpretation of a spinor field as a totally null, generally complex, distribution, but the reality conditions preserving the underlying real metric endows this null geometry with a particular flavour. Thus, the foliating spinor equation (1.3), which corresponds to a shearfree congruence of null geodesics in Lorentzian geometry, must be reinterpreted as the integrability of an almost Hermitian structure in Euclidean signature. Consequently, the Goldberg-Sachs theorem admits various versions depending on the metric signature [PH75, AG97, Apo98, GHN10]. The use of complex methods in general relativity sparked much research interest in the study of complex Riemannian manifolds [Ple75], more importantly, anti-self-dual complex spacetimes such as  $\mathcal{H}$ -spaces [New76, KLNT81], and their twistor theory [Pen76].

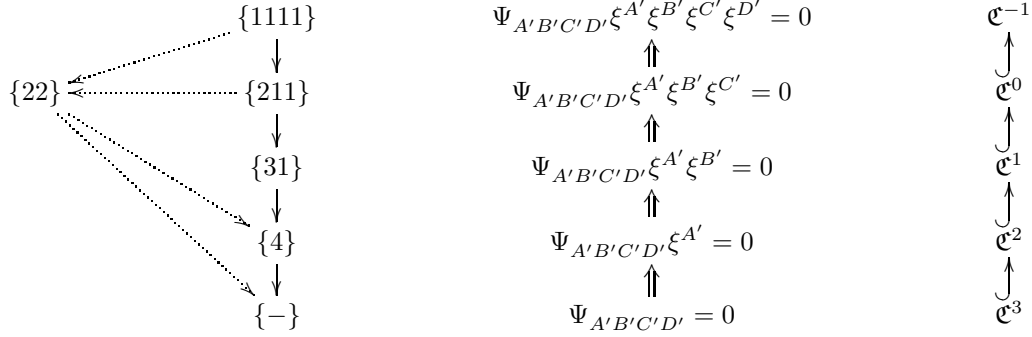
The aims of the present article are

- to give a spinorial classification of the Weyl tensor on a  $2m$ -dimensional complex Riemannian manifold  $(\mathcal{M}, g_{ab})$  along the lines described above;
- to generalise (1.3) and study their relations to spinorial differential equations such as the twistor equation.

For this purpose, we shall build a spinor calculus based on the existence of a single *pure* spinor field on  $\mathcal{M}$ , i.e. a spinor field defining a distribution of totally null  $m$ -planes on  $\mathcal{M}$  [Car81] – this is referred to as an *almost null structure* in [TC12]. This spinor calculus will provide a framework for the classification of curvature tensors and their geometry.

As in four dimensions, the complex Riemannian setting is by no means restrictive and can be applied to real pseudo-Riemannian manifolds by imposing suitable reality conditions. Indeed, previous work has shown [MT10, TC11, TC12] that null distributions, when integrable, underlie the (complex) geometry of a number of higher-dimensional solutions to Einstein's field equations such as the Euclidean and split-signature Kerr-NUT-(A)dS metrics [CLP06, CL08], the Lorentzian Kerr-Schild ansatz [GLPP05] and the black ring [ER02]. However, one advantage of the complex Riemannian case is that the geometry of an almost null structure is made particularly transparent. Similar considerations apply in split signature when spinor fields are taken to be real, and it is in this setting that (integrable or not) almost null structures have received much attention in relation to the conformal geometry of differential equations and Cartan geometry, as can be seen notably in references [Nur05, Bry06, HS11].

To make the aims of our paper more precise, we return to the four-dimensional case for a moment. We first note that the original Petrov-Penrose classification was formulated in a completely invariant way, in the sense that the Petrov-Penrose types are defined by the vanishing of curvature invariants and covariants, which *determine* the principal spinors of  $\Psi_{A'B'C'D'}$ . Our approach will be otherwise, and we shall for convenience *assume* the existence of a principal spinor  $\xi^{A'}$  of  $\Psi_{A'B'C'D'}$ , and think of the Petrov-Penrose classification as being defined *with respect to*  $\xi^{A'}$ . Then the relation between the various Petrov-Penrose types, i.e. the ‘Penrose diagram’ given in the left-hand-side column below, and the algebraic degeneracy  $\Psi_{A'B'C'D'}$  with respect to  $\xi^{A'}$  can be represented schematically as



where the right-hand-side column is part of the filtration

$$\{0\} =: \mathfrak{C}^3 \subset \mathfrak{C}^2 \subset \mathfrak{C}^1 \subset \mathfrak{C}^0 \subset \mathfrak{C}^{-1} \subset \mathfrak{C}^{-2} := \mathfrak{C}. \quad (1.4)$$

on the space  $\mathfrak{C}$  of self-dual Weyl tensors: each vector subspace  $\mathfrak{C}^i$  is defined by the algebraic condition found on the middle column, e.g.  $\mathfrak{C}^1 := \{\Psi_{A'B'C'D'} \in \mathfrak{C} : \Psi_{A'B'C'D'}\xi^{A'}\xi^{B'} = 0\}$ . If  $\Psi_{A'B'C'D'}$  lies in  $\mathfrak{C}$ , but does not degenerate to  $\mathfrak{C}^{-1}$ , then  $\xi^{A'}$  cannot be a principal spinor for it. The filtration (1.4) is manifestly invariant under the subgroup  $P$  of  $\mathrm{SL}(2, \mathbb{C})$  which stabilises the spinor  $\xi^{A'}$ . Such a  $P$  is known as a *Lie parabolic subgroup* of  $\mathrm{SL}(2, \mathbb{C})$ , and such Lie groups form a fundamental ingredient in many geometric constructions [BE89, FH91, ČS09].

It was shown in [TC12] that in  $2m$  dimensions, with  $m > 2$ , if  $\mathfrak{C}$  is now taken to be the space of Weyl tensors, then the filtration (1.4) provides a natural spinorial classification of the Weyl tensor, within which a higher-dimensional generalisation of the complex Goldberg-Sachs theorem can be formulated. This classification is however ‘coarse’ in the following sense: the Lie parabolic subgroup  $P$  of the complex special orthogonal group  $\mathrm{SO}(2m, \mathbb{C})$  stabilising a pure spinor  $\xi^{A'}$  admits the Levy decomposition  $P = G_0 \ltimes P_+$  where  $G_0$  is a reductive group isomorphic to the general linear group  $\mathrm{GL}(m, \mathbb{C})$  and  $P_+$  is nilpotent. Each summand in the associated graded vector space  $\mathrm{gr}(\mathfrak{C}) = \bigoplus_{i=-2}^2 \mathfrak{C}^i / \mathfrak{C}^{i+1}$  is a  $P$ -module on which  $P_+$  acts trivially, and  $G_0$  acts reducibly. It follows that each  $P$ -module  $\mathfrak{C}^i / \mathfrak{C}^{i+1}$  can be *refined* in terms of irreducible  $P$ -modules, each linearly isomorphic to an irreducible  $G_0$ -module. The  $P$ -orbit structure of  $\mathfrak{C}$  can ultimately be arranged in the form of a diagram, which in the case of the Weyl tensor, can be thought of as a Penrose diagram.

At this stage, it is noteworthy to mention the antecedents of the present article. Some partial results regarding the classification of the Weyl tensor in six dimensions using ad-hoc spinorial methods were already obtained in the work of Hughston and Jeffries [Hug95, Jef95] based on the concept of principal spinors (see also [HM88]). It is this approach that we shall adopt and develop in the present article. Moreover, some aspects of almost Hermitian geometry can be recovered from the complex Riemannian case, and part of the present work can be thought of as a complex analogue of the classification of curvature tensors given [TV81, FFS94] for almost Hermitian manifolds. This follows from the fact that a projective pure spinor field defines an almost Hermitian structure, the eigenspaces of the complex structure being totally null complex  $m$ -planes.

Another aspect of almost Hermitian geometry that we will draw on is the Gray-Hervella classification [GH80] of  $2m$ -dimensional almost Hermitian manifolds, which consists in classifying the intrinsic torsion of

the  $U(m)$ -structure of the frame bundle where  $U(m)$  denotes the unitary group. This can equivalently be formulated as a measure of the failure of an almost Hermitian manifold to be Kähler: if  $\omega_{ab}$  is the Hermitian 2-form, then at any point  $p$  of the manifold  $\mathcal{M}$ , the Levi-Civita covariant derivative  $\nabla_a \omega_{bc}$  of  $\omega_{ab}$  lies in a direct sum of irreducible  $U(m)$ -modules of the representation  $T_p \mathcal{M} \otimes \mathfrak{u}(m)^\perp$ . Here  $\mathfrak{u}(m)^\perp$  is the orthogonal complement of the Lie algebra  $\mathfrak{u}(m)$  in the Lie algebra  $\mathfrak{so}(2m)$  of  $SO(2m)$ . Each of the resulting classes can be characterised by some differential condition on  $\omega_{bc}$ , the strongest of which being  $\nabla_a \omega_{bc} = 0$ , i.e.  $\mathcal{M}$  is Kähler. Some of these classes can in fact be characterised by a differential condition on the spinor defining  $\omega_{ab}$ . Thus, the foliating spinor equation (1.3) in almost Hermitian geometry is really equivalent to the vanishing of the Nijenhuis tensor.

The spinorial classification of curvature tensors in odd dimensions along the same lines of the present article will be treated separately [TCa]. Different spinorial classifications for five-dimensional Lorentzian manifolds can be found in [DS05, GPGLMG09, God10]. Other higher-dimensional classifications of the Weyl tensor specific to Lorentzian geometry include the one based on null alignment [CMPP04, MCPP05], and alternative higher-dimensional versions of the Goldberg-Sachs theorem in Lorentzian signature have recently been found in that setting [DR09, OPPR12, OPP12]. Extensions of this article in the Lorentzian setting will be presented in [TCb].

The structure of this article is as follows. In section 2 we develop a spinor calculus for a  $2m$ -dimensional complex vector space equipped with a non-degenerate symmetric metric tensor, on which a pure spinor is singled out. The construction is at the same time related to the representation theory of the parabolic Lie subalgebra  $\mathfrak{p}$  of  $\mathfrak{g} := \mathfrak{so}(2m, \mathbb{C})$  which stabilises this spinor. In section 3 we then apply this calculus to the classification of irreducible curvature tensors, in particular, the Weyl tensor and the Cotton-York tensor, for which we give, in Propositions 3.5 and 3.3, higher-dimensional ‘Penrose diagrams’. We carry out the same programme to classify, in Proposition 4.2, the  $\mathfrak{p}$ -module  $\mathfrak{V} \otimes \mathfrak{g}/\mathfrak{p}$ , where  $\mathfrak{V}$  is the standard representation of  $\mathfrak{g}$ .

This algebraic work is then applied in section 5 to classify the intrinsic torsion of the  $P$ -structure of the frame bundle determined by a projective pure spinor field on a  $2m$ -dimensional complex Riemannian manifold  $(\mathcal{M}, g_{ab})$  – here  $P$  is the Lie parabolic subgroup with Lie algebra  $\mathfrak{p}$ . This is tantamount to classifying the Levi-Civita covariant derivative of this spinor field, analogously to the Gray-Hervella classification of almost Hermitian manifolds [GH80]. More precisely, the algebraic classes of Proposition 4.2 measure the extent to which a projective pure spinor field fails to be parallel at a point of  $\mathcal{M}$ , and such a classification provides a generalisation and a refinement of the foliating spinor equation (1.3) from four to even higher dimensions as stated in Proposition 5.4. In section 5.2, we study the relation between solutions to differential equations on pure spinor fields such as the twistor equation, and study their curvature and conformal properties. In particular, Proposition 5.23 and 5.26 give necessary and sufficient conditions for a pure twistor-spinor to be foliating. Finally, we put forward Conjecture 5.29 generalising the complex Goldberg-Sachs theorem of [TC12].

We round up the paper with three appendices. Appendix A contains a brief discussion of spinor calculus in dimensions four and six, in which the results of the main text can be expressed. In appendix B, we have collected a number of ‘transformation rules’ used in conformal geometry. Finally, in appendix C, we describe the irreducibles occurring in the classification of the curvature tensors and intrinsic torsion of a  $P$ -structure in the language of representation theory as formulated in [BE89, ČS09].

## 2 Algebraic background

In this section, we recall the necessary background on Clifford algebras, spinor representations, and the algebraic properties of pure spinors, which can be found in one form or another in the literature [Car81, PR86, BT88, BT89, FH91, HS92]. We shall adopt the abstract index notation of [PR84] for most of this paper, and in particular, the appendix of [PR86], to which the reader is referred to for further details. Standard index-free notation will be used on occasion.

In our approach to the topic, we shall aim to generalise the notion of a principal spinor of a given tensor representation. In other words, we shall essentially derive a ‘spinor’ calculus from a preferred pure spinor,

and emphasise its relation with the representation theory of parabolic Lie subalgebras given in [BE89, ČS09].

## 2.1 Clifford algebras and spinor representations

Let  $\mathfrak{V}$  be a  $n$ -dimensional complex vector space. Elements of  $\mathfrak{V}$  and its dual  $\mathfrak{V}^*$ , will carry upstairs and downstairs lower-case Roman indices respectively, e.g.  $V^a \in \mathfrak{V}$  and  $\alpha_a \in \mathfrak{V}^*$ . This notation extends to tensor products of  $\mathfrak{V}$  and  $\mathfrak{V}^*$ , i.e. we write  $T_{ab}{}^c{}_d$  for an element of  $\otimes^2 \mathfrak{V}^* \otimes \mathfrak{V} \otimes \mathfrak{V}^*$ . We equip  $\mathfrak{V}$  with a non-degenerate symmetric metric tensor  $g_{ab} = g_{(ab)} \in \odot^2 \mathfrak{V}^*$ . Here, as elsewhere, symmetrisation is denoted by round brackets, while skew-symmetrisation by square brackets, e.g.  $\alpha_{abc} = \alpha_{[abc]} \in \wedge^3 \mathfrak{V}^*$ . The metric tensor  $g_{ab}$  together with its inverse  $g^{ab}$  establishes an isomorphism between  $\mathfrak{V}$  and  $\mathfrak{V}^*$ , so that one will lower or raise the indices of tensorial quantities as needed. We shall also make a choice of orientation, i.e. an element of  $\wedge^n \mathfrak{V}$ , and denote the Hodge star operator on  $\wedge^\bullet \mathfrak{V}$  by  $*$ .

The *Clifford algebra*  $\mathcal{Cl}(\mathfrak{V}, g)$  of  $(\mathfrak{V}, g)$  is defined as the quotient algebra  $\otimes^\bullet \mathfrak{V} / \mathfrak{I}$  where  $\mathfrak{I}$  is the ideal generated by elements of the form  $v \otimes v + g(v, v)$ , where  $v \in \mathfrak{V}$ . This implies that  $\mathcal{Cl}(\mathfrak{V}, g)$  is isomorphic to the exterior algebra  $\wedge^\bullet \mathfrak{V}$  as vector spaces, the wedge product of the latter being now replaced by the *Clifford product*  $\cdot : \mathcal{Cl}(\mathfrak{V}, g) \times \mathcal{Cl}(\mathfrak{V}, g) \rightarrow \mathcal{Cl}(\mathfrak{V}, g)$  defined by

$$v \cdot w := v \wedge w - g(v, w),$$

for any  $v$  and  $w$  in  $\mathfrak{V}$  viewed as elements of  $\mathcal{Cl}(\mathfrak{V}, g)$ .

From now on, we assume  $n = 2m$ . It turns out that the Clifford algebra can be realised as the algebra of complex  $2^m \times 2^m$ -matrices, and acts on a  $2^m$ -dimensional complex vector space referred to the *spinor space*  $\mathfrak{S}$  of  $(\mathfrak{V}, g)$ . This spinor space splits as the direct sum

$$\mathfrak{S} = \mathfrak{S}^+ \oplus \mathfrak{S}^-,$$

where  $\mathfrak{S}^+$  and  $\mathfrak{S}^-$  are the  $\pm$ -eigenspaces of the chosen orientation viewed as an element of  $\mathcal{Cl}(\mathfrak{V}, g)$ . The  $2^{m-1}$ -dimensional complex vector spaces  $\mathfrak{S}^+$  and  $\mathfrak{S}^-$  will be referred to as the *positive and negative (chiral) spinor spaces* respectively. It is well-known that these complex vector spaces are irreducible representations of the spin group  $\text{Spin}(2m, \mathbb{C})$ , the double cover of special orthogonal group  $\text{SO}(2m, \mathbb{C})$ .

**Remark 2.1** We can describe these spinor spaces explicitly by choosing a totally null  $m$ -dimensional subspace  $\mathfrak{N} \subset \mathfrak{V}$ , i.e.  $g|_{\mathfrak{N}} = 0$ , and fix a dual  $\mathfrak{N}^*$  of  $\mathfrak{N}$  so that  $\mathfrak{V} \cong \mathfrak{N} \oplus \mathfrak{N}^*$ . Then the vector space  $\wedge^\bullet \mathfrak{N}$  can be turned into the  $\mathcal{Cl}(\mathfrak{V}, g)$ -module  $\mathfrak{S}$  by restricting the Clifford product to it: for any  $\xi \in \wedge^\bullet \mathfrak{S}$ ,  $v \in \mathfrak{N}$  and  $w \in \mathfrak{N}^*$ , the action of  $\mathcal{Cl}(\mathfrak{V}, g)$  on  $\mathfrak{S}$  is given by  $(v, w) \cdot \xi = v \wedge \xi - w \lrcorner \xi$ . With a choice of orientation, we can then make the identifications

$$\mathfrak{S}^+ \cong \wedge^m \mathfrak{N} \oplus \wedge^{m-2} \mathfrak{N} \oplus \dots, \quad \mathfrak{S}^- \cong \wedge^{m-1} \mathfrak{N} \oplus \wedge^{m-3} \mathfrak{N} \oplus \dots$$

Elements of  $\mathfrak{S}^+$ ,  $\mathfrak{S}^-$  will carry upstairs primed, respectively unprimed, upper-case Roman indices, e.g.  $\xi^{A'}$ ,  $\alpha^A$ , respectively, and similarly for their duals  $(\mathfrak{S}^+)^*$  and  $(\mathfrak{S}^-)^*$  with downstairs indices, e.g.  $\eta_{A'}$  and  $\beta_A$  respectively. As we shall be working with the chiral spinor spaces exclusively, it will be convenient to think of the generators of the Clifford algebra  $\mathcal{Cl}(\mathfrak{V}, g)$  in terms of the Van der Waerden  $\gamma$ -matrices  $\gamma_{aA}{}^{B'}$  and  $\gamma_{aA'}{}^B$ , which satisfy the (reduced) Clifford property

$$\gamma_{(aA'}{}^C \gamma_{b)C}{}^{B'} = -g_{ab} \delta_{A'}^{B'}, \quad \gamma_{(aA}{}^{C'} \gamma_{b)C'}{}^B = -g_{ab} \delta_A^B, \quad (2.1)$$

where  $\delta_{A'}^{B'}$  and  $\delta_A^B$  are the identity elements on  $\mathfrak{S}^+$  and  $\mathfrak{S}^-$  respectively. Thus, only skew-symmetrised products of  $\gamma$ -matrices count, and we shall make use of the notational short hand

$$\begin{aligned} \gamma_{a_1 a_2 \dots a_q A}{}^{B'} &:= \gamma_{[a_1 A}{}^{C'_1} \gamma_{a_2 C'_1}{}^{C_2} \dots \gamma_{a_q] C_{q-1}}{}^{B'} , & \gamma_{a_1 a_2 \dots a_q A'}{}^B &:= \gamma_{[a_1 A'}{}^{C_1} \gamma_{a_2 C_1}{}^{C'_2} \dots \gamma_{a_q] C'_{q-1}}{}^B , \\ \gamma_{a_1 a_2 \dots a_p A}{}^B &:= \gamma_{[a_1 A}{}^{C'_1} \gamma_{a_2 C'_1}{}^{C_2} \dots \gamma_{a_p] C_p}{}^B , & \gamma_{a_1 a_2 \dots a_p A'}{}^{B'} &:= \gamma_{[a_1 A'}{}^{C_1} \gamma_{a_2 C_1}{}^{C'_2} \dots \gamma_{a_p] C_{p-1}}{}^{B'} , \end{aligned} \quad (2.2)$$

where  $p$  is even and  $q$  is odd. These matrices give us an explicit realisation of the isomorphism  $\mathcal{C}\ell(\mathfrak{V}, g_{ab}) \cong \bigwedge^{\bullet} \mathfrak{V}$  as vector spaces.

The spinor space  $\mathfrak{S}$  and its dual  $\mathfrak{S}^*$  are equipped with  $\text{Spin}(2m, \mathbb{C})$ -invariant non-degenerate bilinear forms, which realise the isomorphisms

$$\begin{aligned} \mathfrak{S}^{\pm} &\cong (\mathfrak{S}^{\pm})^*, & m \text{ even}, \\ \mathfrak{S}^{\pm} &\cong (\mathfrak{S}^{\mp})^*, & m \text{ odd}. \end{aligned} \quad (2.3)$$

One can in effect raise or lower spinor indices,<sup>2</sup> which may be become primed or unprimed, according to the parity of  $m$ . In particular, the  $\gamma$ -matrices (2.2) can now be viewed as bilinear maps from  $\mathfrak{S}^{\pm} \times \mathfrak{S}^{\pm}$  or  $\mathfrak{S}^{\pm} \times \mathfrak{S}^{\mp}$  to  $\bigwedge^p \mathfrak{V}$  for some  $p$ . Again, these satisfy some symmetry properties depending on  $m$  and  $p$  as explained in [PR86]. Our treatment will be overwhelmingly dimension independent, and for this reason, we shall avoid making use of the  $\text{Spin}(2m, \mathbb{C})$ -invariant bilinear forms  $\mathfrak{S}$ . It suffices to say that when  $p = m$ , the  $\gamma$ -matrices always take the form  $\gamma_{a_1 a_2 \dots a_m}{}^{A' B'}$  and  $\gamma_{a_1 a_2 \dots a_m}{}^{AB}$  and are symmetric in their spinor indices. More precisely, we have injective maps

$$\gamma_{a_1 a_2 \dots a_m}{}^{A' B'} : \bigwedge_+^m \mathfrak{V} \rightarrow \odot^2 \mathfrak{S}^+, \quad \gamma_{a_1 a_2 \dots a_m}{}^{AB} : \bigwedge_-^m \mathfrak{V} \rightarrow \odot^2 \mathfrak{S}^-,$$

or dually, surjective maps

$$\gamma_{a_1 a_2 \dots a_m A' B'} : \bigwedge_+^m \mathfrak{V}^* \leftarrow \odot^2 \mathfrak{S}^+, \quad \gamma_{a_1 a_2 \dots a_m AB} : \bigwedge_-^m \mathfrak{V}^* \leftarrow \odot^2 \mathfrak{S}^-,$$

where  $\bigwedge_+^m \mathfrak{V}$  and  $\bigwedge_-^m \mathfrak{V}$  are the irreducible  $\text{SO}(2m, \mathbb{C})$ -modules of self-dual and anti-self-dual  $m$ -vectors respectively, i.e. the eigenspaces of the Hodge star operator  $*$  on  $\bigwedge^m \mathfrak{V}$ .

Before we delve into the topic of pure spinors, we state without proof the following technical lemma for later use.<sup>3</sup>

**Lemma 2.2** *When  $m - p$  is even,*

$$\begin{aligned} \gamma_{a A'}{}^B \gamma_{b_1 \dots b_p B D} \gamma_{c C'}{}^D &= (-1)^m \left( \gamma_{c a b_1 \dots b_p A' C'} + g_{ca} \gamma_{b_1 \dots b_{p-1} b_p A' C'} \right. \\ &\quad \left. - 2p g_{[b_1 | (a} \gamma_{c) | b_2 \dots b_p] A' C'} + p(p+1) g_{a [b_1} g_{| c | b_2} \gamma_{b_3 \dots b_{p-1} b_p] A' C'} \right) \end{aligned}$$

*In particular,*

$$\gamma_{A'}{}^a \gamma_{b_1 \dots b_p B D} \gamma_{a C'}{}^D = (-1)^m 2(m-p) \gamma_{b_1 \dots b_p A' C'}.$$

*When  $m - p$  is odd,*

$$\begin{aligned} \gamma_{a A'}{}^B \gamma_{b_1 \dots b_p B D} \gamma_{c C'}{}^{D'} &= (-1)^{m-1} \left( \gamma_{c a b_1 \dots b_p A' C} + g_{ca} \gamma_{b_1 \dots b_{p-1} b_p A' C} \right. \\ &\quad \left. - 2p g_{[b_1 | (a} \gamma_{c) | b_2 \dots b_p] A' C} + p(p+1) g_{a [b_1} g_{| c | b_2} \gamma_{b_3 \dots b_{p-1} b_p] A' C} \right) \end{aligned}$$

*In particular,*

$$\gamma_{A'}{}^a \gamma_{b_1 \dots b_p B D} \gamma_{a C}{}^{D'} = (-1)^{m-1} 2(m-p) \gamma_{b_1 \dots b_p A' C}.$$

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<sup>2</sup>In [PR86], these  $\text{Spin}(2m, \mathbb{C})$ -invariant bilinear forms are denoted  $\varepsilon_{A' B'}$ ,  $\varepsilon_{AB}$  when  $m$  is even, and  $\varepsilon_{A' B}$  and  $\varepsilon_{A B'}$  when  $m$  is odd, and similarly for the dual spinor spaces. Their symmetry properties, which depend on  $m$ , are given in [PR86]

<sup>3</sup>The proof of the lemma makes use of the symmetry properties of the  $\gamma$ -matrices, which can be found in [PR86].

## 2.2 Pure spinors

Let  $\xi^{A'}$  be a non-zero spinor in  $\mathfrak{S}^+$ , and consider the map

$$\xi_a^A := \xi^{B'} \gamma_{aB'}^A : \mathfrak{V} \rightarrow \mathfrak{S}^-.$$

By (2.1), the kernel of  $\xi_a^A : \mathfrak{V} \rightarrow \mathfrak{S}^-$  is totally null, i.e. for any  $X^a, Y^a \in \ker \xi_a^A : \mathfrak{V} \rightarrow \mathfrak{S}^-$ ,  $g_{ab}X^aY^b = 0$ .

**Definition 2.3** A non-zero (positive) spinor  $\xi^{A'}$  is said to be *pure* if the kernel of  $\xi_a^A : \mathfrak{V} \rightarrow \mathfrak{S}^-$  is  $m$ -dimensional.

Clearly, the purity property is invariant under rescaling of  $\xi^{A'}$ , and applies equally to negative spinors. In fact, one can show that a pure spinor is necessarily chiral. We shall refer to the complex span of a pure spinor as a *projective spinor*.

**Definition 2.4** A *null structure* on  $\mathfrak{V}$  is a totally null  $m$ -dimensional vector subspace of  $\mathfrak{V}$ . A null structure defined by a positive, respectively negative, spinor is called an  $\alpha$ -*plane*, respectively a  $\beta$ -*plane*.

**Remark 2.5** That non-zero pure spinors exist follows from the description, given in Remark 2.1 of  $\mathfrak{S}$  as the vector space  $\wedge^\bullet \mathfrak{N}$  viewed as a  $\mathcal{C}\ell(\mathfrak{V}, g)$ -module: the spinor  $\xi$  lying in  $\wedge^m \mathfrak{N}$  is pure since  $(v, 0) \cdot \xi = 0$  for all  $v \in \mathfrak{N}$ .

From now on, we assume that  $\xi^{A'}$  is pure. For convenience, we set

$$\mathfrak{S}^{\frac{m}{4}} := \langle \xi^{A'} \rangle, \quad \mathfrak{S}^{\frac{m-2}{4}} := \text{im } \xi_a^A : \mathfrak{V} \rightarrow \mathfrak{S}^-, \quad \mathfrak{V}^{-\frac{1}{2}} := \mathfrak{V}, \quad \mathfrak{V}^{\frac{1}{2}} := \ker \xi_a^A : \mathfrak{V} \rightarrow \mathfrak{S}^-,$$

so that one can express the  $\alpha$ -plane associated to  $\xi^{A'}$  as the filtration

$$\{0\} =: \mathfrak{V}^{\frac{3}{2}} \subset \mathfrak{V}^{\frac{1}{2}} \subset \mathfrak{V}^{-\frac{1}{2}}. \quad (2.4)$$

The full meaning of this notation, borrowed from [ČS09], will be explained in the course of this section. For the moment, the reader should think of these numerical indices as homogeneity degrees. By definition, these vector spaces are related via the isomorphism

$$\left( \mathfrak{V}^{-\frac{1}{2}} / \mathfrak{V}^{\frac{1}{2}} \right) \otimes \mathfrak{S}^{\frac{m}{4}} \cong \mathfrak{S}^{\frac{m-2}{4}}. \quad (2.5)$$

While the factor  $\otimes \mathfrak{S}^{\frac{m}{4}}$  on the LHS of (2.5) may appear notationally redundant, it nonetheless balances the degrees on each side of (2.5), i.e.  $-\frac{1}{2} + \frac{m}{4} = \frac{m-2}{4}$ . From (2.5), it is also clear that  $\mathfrak{S}^{\frac{m-2}{4}}$  is an  $m$ -dimensional subspace of  $\mathfrak{S}^-$ .

With a slight abuse of notation, we can also think of the map  $\xi_a^A$  dually as  $\xi_a^A : \mathfrak{V}^* \leftarrow (\mathfrak{S}^-)^*$  so that the dual counterpart of (2.5) is given by

$$\mathfrak{V}^{\frac{1}{2}} \cong \mathfrak{S}^{\frac{m}{4}} \otimes \left( \mathfrak{S}^{-\frac{m-2}{4}} / \mathfrak{S}^{-\frac{m-6}{4}} \right), \quad (2.6)$$

where we have defined

$$\mathfrak{S}^{-\frac{m-2}{4}} := (\mathfrak{S}^-)^*, \quad \mathfrak{S}^{-\frac{m-6}{4}} := \ker \xi_a^A : \mathfrak{V}^* \leftarrow (\mathfrak{S}^-)^*,$$

and made use of  $\mathfrak{V}^{\frac{1}{2}} \cong \left( \mathfrak{V}^{-\frac{1}{2}} / \mathfrak{V}^{\frac{1}{2}} \right)^*$ . Isomorphism (2.6) can be expressed concretely as follows.

**Lemma 2.6** A non-zero vector  $V^a$  is an element of  $\mathfrak{V}^{\frac{1}{2}}$  if and only if  $V^a = \xi^{aB} v_B$  for some non-zero spinor  $v_A$  in  $\mathfrak{S}^{-\frac{m-2}{4}} / \mathfrak{S}^{-\frac{m-6}{4}}$ .

Since  $\mathfrak{V}^{\frac{1}{2}}$  is a totally null  $m$ -dimensional vector subspace, we can now conclude

**Lemma 2.7** *A (positive) spinor  $\xi^{A'}$  is pure if and only if it satisfies*

$$\xi^{aA} \xi_a^{B'} = 0. \quad (2.7)$$

This characterisation of pure spinors can already be found in the work of [HM88]. In fact, applying Lemma 2.2 to Lemma 2.7, one recovers the following well-known characterisation of pure spinors due to Cartan.

**Proposition 2.8 (Cartan [Car81])** *A spinor  $\xi^{A'}$  is pure if and only if it satisfies*

$$\gamma_{a_1 \dots a_p A' B'} \xi^{A'} \xi^{B'} = 0, \quad \text{for all } p < m, \quad m - p \equiv 0 \pmod{4}. \quad (2.8)$$

We shall refer to both equations (2.7) and (2.8) as the *purity conditions* of a spinor  $\xi^{A'}$ . These are vacuous for  $m \leq 3$ , and so all spinors are pure when  $m \leq 3$ . When  $m > 3$ , these conditions are quadratic on the components of a pure spinor.

Proposition 2.8 tells us that the only non-trivial irreducible component of the tensor product  $\xi^{A'} \xi^{B'}$  of a pure spinor  $\xi^{A'}$  lies in  $\wedge^m_+ \mathfrak{V}$ . In fact, the self-dual  $m$ -form

$$\phi_{a_1 \dots a_m} := \gamma_{a_1 \dots a_m A' B'} \xi^{A'} \xi^{B'}$$

annihilates  $\mathfrak{V}^{\frac{1}{2}}$ , i.e.  $\xi^{a_1 A} \phi_{a_1 a_2 \dots a_m} = 0$ . In particular, it must be simple (or decomposable), i.e.

$$\phi_{a_1 \dots a_m} = \xi_{a_1}^{A_1} \dots \xi_{a_m}^{A_m} \varepsilon_{A_1 \dots A_m} \in \wedge^m \mathfrak{V}^{\frac{1}{2}},$$

for some  $\varepsilon_{A_1 \dots A_m} \in \wedge^m \left( \mathfrak{S}^{-\frac{m-2}{4}} / \mathfrak{S}^{-\frac{m-6}{4}} \right)$ . For this reason,  $\alpha$ -planes are self-dual, and similarly,  $\beta$ -planes anti-self-dual.

It is often more convenient to eliminate the quotient vector spaces in the isomorphisms (2.5) and (2.6) in favour of splittings adapted to them. We first set

$$\mathfrak{S}^{-\frac{m}{4}} := (\mathfrak{S}^+)^*, \quad \mathfrak{S}^{-\frac{m-4}{4}} := \ker \xi^{A'} : \mathbb{C} \leftarrow (\mathfrak{S}^+)^*.$$

We now let  $\eta_{A'} \in \mathfrak{S}^{-\frac{m}{4}}$  such that  $\xi^{A'} \eta_{A'} \neq 0$ , i.e.  $\eta_{A'}$  is not in  $\mathfrak{S}^{-\frac{m-4}{4}}$ . This distinguishes a one-dimensional subspace  $\langle \eta_{A'} \rangle =: \mathfrak{S}_{-\frac{m}{4}} \subset \mathfrak{S}^{-\frac{m}{4}}$  complementary to  $\mathfrak{S}^{-\frac{m-4}{4}}$ . We associate to  $\eta_{A'}$  the map  $\eta_{aA} := \gamma_{aA}^{B'} \eta_{B'} : \mathfrak{V} \rightarrow (\mathfrak{S}^-)^*$ , and for future use, we define

$$\mathfrak{S}_{-\frac{m-2}{4}} := \text{im } \eta_{aA} : \mathfrak{V} \rightarrow (\mathfrak{S}^-)^*, \quad \mathfrak{V}_{-\frac{1}{2}} := \text{im } \eta^a_A : \mathfrak{V} \leftarrow \mathfrak{S}^-.$$

With no loss of generality, we fix the scales of  $\xi^{A'}$  and  $\eta_{A'}$  so that  $\xi^{A'} \eta_{A'} = -\frac{1}{2}$ . By the Clifford property (2.1) one can show that the map

$$I_B^A := \eta_{aB} \xi^{aA} : \mathfrak{S}^- \rightarrow \mathfrak{S}^- \quad (2.9)$$

is idempotent with trace  $I_A^A = m$ . Thus,  $I_B^A$  must be the identity on  $\mathfrak{S}_{\frac{m-2}{4}} := \mathfrak{S}^{-\frac{m-2}{4}}$ , or dually, on  $\mathfrak{S}_{-\frac{m-2}{4}}$ . In particular,  $\eta_{A'}$  must be pure with associated null structure  $\mathfrak{V}_{-\frac{1}{2}}$ , which is dual to  $\mathfrak{V}_{\frac{1}{2}} := \mathfrak{V}^{\frac{1}{2}}$  via the metric identification  $\mathfrak{V} \cong \mathfrak{V}^*$ , and is thus linearly isomorphic to  $\mathfrak{V}^{-\frac{1}{2}} / \mathfrak{V}^{\frac{1}{2}}$ . This yields a direct sum decomposition

$$\mathfrak{V} \cong \mathfrak{V}_{-\frac{1}{2}} \oplus \mathfrak{V}_{\frac{1}{2}}, \quad (2.10)$$

adapted to the filtration (2.4). In fact, it is not difficult to see that conversely, any splitting of  $\mathfrak{V}$  as in (2.10) determines a spinor  $\eta_{A'}$  dual to  $\xi^{A'}$ . Further, using the identifications (2.3),  $\mathfrak{V}_{-\frac{1}{2}}$  is an  $\alpha$ -plane, respectively, a  $\beta$ -plane, when  $m$  is even, respectively, odd.

Finally, we can apply Lemma 2.6 to  $\eta_{A'}$  (or any pure spinor for that matter) in the sense that any vector  $V^a$  in  $\mathfrak{V}_{-\frac{1}{2}}$  takes the form  $V^a = \eta^a_A v^A$  for some spinor  $v^A$  in  $\mathfrak{S}_{\frac{m-2}{4}}$ .

We end the section with a proposition characterising the intersections of  $\alpha$ -planes and  $\beta$ -planes in terms of spinors.



**Proposition 2.9** Any spinor  $\beta^A$  in  $\mathfrak{S}^{\frac{m-2}{4}}$  is a pure spinor. Further, the  $\beta$ -plane defined by  $\beta^A$  intersects the  $\alpha$ -plane defined by a generator  $\xi^{A'}$  of  $\mathfrak{S}^{\frac{m}{4}}$  in a totally null  $(m-1)$ -plane. This is algebraically equivalent to  $\beta^A$  and  $\xi^{A'}$  satisfying

$$\xi^{aA} \beta_a^{B'} = -2\xi^{B'} \beta^A, \quad (2.11)$$

or equivalently,

$$\gamma_{a_1 a_2 \dots a_p A' B} \xi^{A'} \beta^B = 0, \quad \text{for all } p < m-1, m-p \equiv 1 \pmod{2}. \quad (2.12)$$

Finally, the  $\beta$ -planes of any distinct (projective) spinors,  $\beta^B$  and  $\rho^A$  say, in  $\mathfrak{S}^{\frac{m-2}{4}}$  intersect in an  $(m-2)$ -dimensional totally null vector subspace. This is algebraically equivalent to  $\beta^A$  and  $\rho^A$  satisfying

$$\beta^{a(A'} \rho_a^{B')} = 0, \quad (2.13)$$

or equivalently,

$$\gamma_{a_1 a_2 \dots a_p AB} \beta^A \rho^B = 0, \quad \text{for all } p < m-2, m-p \equiv 0 \pmod{2}. \quad (2.14)$$

*Proof.* The proposition is really an application of the standard results of [Car81], but one can give an explicit argument using our spinor calculus. Let  $\beta^A$  be a projective spinor in  $\mathfrak{S}^{\frac{m-2}{4}}$  so that  $\beta^A = b^a \xi_a^A$  for some  $b^a$  not in the kernel of  $\xi_a^A$ . Thus  $b^a$  must be null and annihilate  $\beta^A$ . Then,

$$\beta^{aA'} \beta_a^{B'} = b^a b^b \gamma_{aA}^{A'} \gamma_{bB}^{B'} (\xi^{cA} \xi_c^B) + 4b^a \beta_a^{(A'} \xi^{B')} + 4b_a b^a \xi^{A'} \xi^{B'} = 0,$$

which shows that  $\beta^A$  is pure.

A similar computation shows that the  $\beta$ -plane associated to  $\beta^A$  is spanned by  $b^a$  and any  $m-1$  vectors in  $\mathfrak{V}_{\frac{1}{2}}$  orthogonal to  $b^a$ . Deriving the algebraic condition (2.11) is purely computational and its equivalence to (2.12) can be shown by means of Lemma 2.2.

Now, choose a projective spinor  $\rho^A$  in  $\mathfrak{S}^{\frac{m-2}{4}}$  distinct from  $\beta^A$ , i.e.  $\beta^{[A} \rho^{B]} \neq 0$ . Then  $\rho^A$  is a pure spinor such that  $\rho^A = r^a \xi_a^A$  for some  $r^a$  not in the kernel of  $\xi_a^A$ , and with  $b^{[a} r^{b]} \neq 0$ . The  $\beta$ -plane associated to  $\rho^A$  is spanned by  $r^a$  and any  $m-1$  vectors in  $\mathfrak{V}_{\frac{1}{2}}$  orthogonal to  $r^a$ . Since  $b^a$  and  $r^a$  are not proportional to each other, the  $\beta$ -planes defined by  $\beta^A$  and  $\rho^A$  must have an  $(m-2)$ -dimensional vector subspace of  $\mathfrak{V}_{\frac{1}{2}}$  in common, orthogonal to both  $b^a$  and  $r^a$ . Now, by the first part of the proposition, since  $\mathfrak{S}^{\frac{m-2}{4}}$  is a vector space of pure spinors, the sum of  $\beta^A$  and  $\rho^B$  is also a pure spinor. By polarisation, we then obtain the algebraic condition (2.13) for their intersection. This can be shown to be equivalent to (2.14) by applying Lemma 2.2.  $\square$

**Remark 2.10** The second part of Proposition 2.9 is an articulation of a standard theorem (see e.g. [BT89] and references therein) which states that a sufficient and necessary condition for the sum of two pure spinors to be pure is that their respective totally null  $m$ -planes intersect in a totally null  $(m-2)$ -plane.

### 2.3 Classification of the Lie algebra $\mathfrak{so}(2m, \mathbb{C})$ for $m > 2$

**Filtration** We now turn to the decomposition of the Lie algebra  $\mathfrak{g} := \mathfrak{so}(2m, \mathbb{C})$ , which we shall identify with the space  $\wedge^2 \mathfrak{V}^*$  of 2-forms by means of the metric tensor  $g_{ab}$ . For definiteness, we assume  $m > 2$ , and refer the reader to appendix A.1 for the case  $m = 2$ . Defining, for any  $\phi_{ab} \in \mathfrak{g}$ ,

$${}^{\mathfrak{g}}\Pi_{-1}(\phi) := \xi^{aA} \xi^{bB} \phi_{ab}, \quad {}^{\mathfrak{g}}\Pi_0(\phi) := \xi^{bA} \phi_{ba}, \quad (2.15)$$

and setting

$$\mathfrak{g}^i := \{\phi_{ab} \in \mathfrak{g} : {}^{\mathfrak{g}}\Pi_{i-1}(\phi) = 0\}, \quad (2.16)$$

for  $i = 0, 1$ , we obtain a filtration of vector subspaces

$$\{0\} =: \mathfrak{g}^2 \subset \mathfrak{g}^1 \subset \mathfrak{g}^0 \subset \mathfrak{g}^{-1} := \mathfrak{g} \quad (2.17)$$

of  $\mathfrak{g}$ . In fact, as can easily be checked from the definitions (2.16),  $\mathfrak{g}$  is a *filtered Lie algebra*, in the sense that the Lie bracket  $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  is compatible with the filtration on  $\mathfrak{g}$ , i.e.  $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$ , with the convention  $\mathfrak{g}^i = \{0\}$  for  $i \geq 2$ , and  $\mathfrak{g}^i = \mathfrak{g}$  for all  $i \leq -1$ .

Applying Lemma 2.6 to any element of  $\mathfrak{g}^i$ , for  $i = 0, 1$ , yields

**Lemma 2.11** *Let  $\phi_{ab}$  be a non-zero element in  $\mathfrak{g}$ . Then*

- $\phi_{ab} \in \mathfrak{g}^0$  if and only if  $\phi_{ab} = \xi_{[a}^A \phi_{Ab]}$  for some  $\phi_{Ab} \in \left( \mathfrak{S}^{-\frac{m-2}{4}} / \mathfrak{S}^{-\frac{m-6}{4}} \right) \otimes \mathfrak{V}^{-\frac{1}{2}}$ ;
- $\phi_{ab} \in \mathfrak{g}^1$  if and only if  $\phi_{ab} = \xi_a^C \xi_b^D \phi_{CD}$  for some  $\phi_{AB} = \phi_{[AB]} \in \wedge^2 \left( \mathfrak{S}^{-\frac{m-2}{4}} / \mathfrak{S}^{-\frac{m-6}{4}} \right)$ .

**Remark 2.12** The degeneracy of a spinor representative  $\phi_{ab}$  from  $\mathfrak{g}^0$  to  $\mathfrak{g}^1$  is given by  $\phi_{Ab} = \phi_{AB} \xi_b^B$ .

**Lemma 2.13** *The Lie subalgebra  $\mathfrak{g}^0$  is the stabiliser of  $\xi^{A'}$  in the sense that*

$$\phi_{ab} \gamma^{ab}{}_{B'}{}^{A'} \xi^{B'} \propto \xi^{A'}.$$

*Proof.* From the identity  $\xi^{aA} \xi^{bB} \phi_{ab} = -\frac{1}{4} \phi_{ab} \gamma^{ab}{}_{B'}{}^{A'} \xi^{B'} \gamma^c{}_{A'}{}^C \xi_c^E$ , it follows that the stabiliser of  $\xi^{A'}$  is contained in  $\mathfrak{g}^0$ . In fact, it must be  $\mathfrak{g}^0$  by the decomposition given in Lemma 2.11.  $\square$

The Lie subalgebra  $\mathfrak{p} := \mathfrak{g}^0$  is a *Lie parabolic subalgebra* of  $\mathfrak{so}(2m, \mathbb{C})$ . From the Lie bracket commutation relation of  $\mathfrak{g}^i$ , each vector subspace  $\mathfrak{g}^i$  is a  $\mathfrak{p}$ -module.

**Associated graded vector space** To refine the filtration (2.17), we introduce the associated graded vector space  $\text{gr}(\mathfrak{g}) = \bigoplus_{i=-1}^1 \text{gr}_i(\mathfrak{g})$  where  $\text{gr}_i(\mathfrak{g}) := \mathfrak{g}^i / \mathfrak{g}^{i+1}$ . Define

$$\xi_{ab}{}^{A'} := \xi^{B'} \gamma_{abB'}{}^{A'} : \wedge^2 \mathfrak{V} \rightarrow \mathfrak{S}^+, \quad \mathfrak{S}^{\frac{m-4}{4}} := \text{im } \xi_{ab}{}^{A'} : \wedge^2 \mathfrak{V} \rightarrow \mathfrak{S}^+.$$

We note that one must have

$$\mathfrak{S}^{\frac{m}{4}} \subset \mathfrak{S}^{\frac{m-4}{4}} \quad (2.18)$$

since for any  $\lambda \in \mathbb{C}$  there exist  $X^a \in \mathfrak{V}^{\frac{1}{2}}$  and  $Y^b \in \mathfrak{V}^{-\frac{1}{2}}$  such that  $X^a Y_a = \lambda$  and  $\lambda \xi^{A'} = X^a Y^b g_{ab} \xi^{A'} = X^a Y^b \xi_{ab}{}^{A'}$  by the Clifford property (2.1). Since  $\xi^{aA} \xi_{ab}{}^{B'} = \xi^{B'} \xi_b^A$  we have that the image of the restriction of  $\xi_{ab}{}^{A'}$  to  $\mathfrak{g}^0$  must be  $\mathfrak{S}^{\frac{m}{4}}$  from which it follows that

$$(\mathfrak{g}^{-1} / \mathfrak{g}^0) \otimes \mathfrak{S}^{\frac{m}{4}} \cong \mathfrak{S}^{\frac{m-4}{4}} / \mathfrak{S}^{\frac{m}{4}}. \quad (2.19)$$

Again, the map  $\xi_{ab}{}^{A'}$  can be viewed dually, and we set

$$\mathfrak{S}^{-\frac{m-8}{4}} := \ker \xi_{ab}{}^{A'} : \wedge^2 \mathfrak{V}^* \leftarrow (\mathfrak{S}^+)^*.$$

Note that if  $\alpha_{A'} \in \mathfrak{S}^{-\frac{m-8}{4}}$ , then we have  $0 = \xi_{ab}{}^{A'} \alpha_{A'} = \xi_a^B \gamma_{bB}{}^{A'} \alpha_{A'} + g_{ab} \xi^{A'} \alpha_{A'}$ . Now, contracting this equation with any  $V^a \in \mathfrak{V}^{\frac{1}{2}}$  and  $W^b \in \mathfrak{V}$  such that  $V^a W_a \neq 0$ , we get  $0 = V^a W_a \xi^{A'} \alpha_{A'}$ . It now follows that

$$\mathfrak{S}^{-\frac{m-8}{4}} \subset \mathfrak{S}^{-\frac{m-4}{4}} \subset \mathfrak{S}^{-\frac{m}{4}}. \quad (2.20)$$

The dual of isomorphism (2.19) can thus be written as

$$\mathfrak{g}^1 \cong \mathfrak{S}^{\frac{m}{4}} \otimes \left( \mathfrak{S}^{-\frac{m-4}{4}} / \mathfrak{S}^{-\frac{m-8}{4}} \right). \quad (2.21)$$

Let us now adorn the image of the map  ${}^{\mathfrak{g}}\Pi_0$  with free indices, i.e. write  ${}^{\mathfrak{g}}\Pi_0(\phi)^{aA}$ , and observe that it can be decomposed further as  ${}^{\mathfrak{g}}\Pi_0(\phi)^{aA} = {}^{\mathfrak{g}}\Pi_0^1(\phi)^{aA} - \frac{1}{n} {}^{\mathfrak{g}}\Pi_0^0(\phi)^{B'} \gamma^a_{B'}{}^A$  where, for any  $\phi_{ab} \in \mathfrak{g}$ ,

$${}^{\mathfrak{g}}\Pi_0^0(\phi)^{A'} := \xi^{abA'} \phi_{ab}, \quad {}^{\mathfrak{g}}\Pi_0^1(\phi)^A{}_b := \xi^{cA} \phi_{cb} + \frac{1}{n} \gamma_{bC'}{}^A \xi^{cdC'} \phi_{cd}. \quad (2.22)$$

Clearly, any element  $\phi_{ab}$  of  $\mathfrak{g}^1$  will satisfy  ${}^{\mathfrak{g}}\Pi_0^1(\phi)^A{}_c = {}^{\mathfrak{g}}\Pi_0^0(\phi)^{A'} = 0$ . Further, since  ${}^{\mathfrak{g}}\Pi_0^1(\phi)^A{}_c$  and  ${}^{\mathfrak{g}}\Pi_0^0(\phi)^{A'}$  are complementary to one another in the sense that  ${}^{\mathfrak{g}}\Pi_0^1(\phi)^A{}_c \gamma^c{}_A{}^{B'} = 0$ , there is a direct sum decomposition

$$\mathfrak{g}^0 / \mathfrak{g}^1 \cong \mathfrak{g}_0^0 \oplus \mathfrak{g}_0^1,$$

where we have defined vector subspaces

$$\mathfrak{g}_0^0 := \{ \phi_{ab} \in \mathfrak{g}^0 : {}^{\mathfrak{g}}\Pi_0^1(\phi) = 0 \} / \mathfrak{g}^1, \quad \mathfrak{g}_0^1 := \{ \phi_{ab} \in \mathfrak{g}^0 : {}^{\mathfrak{g}}\Pi_0^0(\phi) = 0 \} / \mathfrak{g}^1,$$

of  $\mathfrak{g}^0 / \mathfrak{g}^1$ . For future use, we also set  $\mathfrak{g}_1^0 := \mathfrak{g}^1$ ,  $\mathfrak{g}_{-1}^0 := \mathfrak{g}^{-1} / \mathfrak{g}^0$ , and  ${}^{\mathfrak{g}}\Pi_{-1}^0 := {}^{\mathfrak{g}}\Pi_{-1}$ . By construction, each summand  $\mathfrak{g}_i^j$  of  $\text{gr}(\mathfrak{g})$  are  $\mathfrak{p}$ -module.

**Grading** Recall that the choice of a spinor  $\eta_{A'}$  dual  $\xi^{A'}$  induces a direct sum decomposition of  $\mathfrak{V}$  adapted to the  $\alpha$ -plane defined by  $\xi^{A'}$ . This clearly extends to the Lie algebra  $\mathfrak{g}$ , and in fact endows  $\mathfrak{g}$  with the structure of a  $|1|$ -graded Lie algebra, i.e.

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1, \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}.$$

where  $\mathfrak{g}_i \subset \mathfrak{g}^i$  are complementary to  $\mathfrak{g}^{i+1}$ , for each  $i = -1, 0, 1$ ,  $\mathfrak{g}_i := \{0\}$  when  $|i| > 2$  for convenience. Explicitly, we have

$$\mathfrak{g}_{-1} \cong \wedge^2 \mathfrak{V}_{-\frac{1}{2}}, \quad \mathfrak{g}_0 \cong \mathfrak{V}_{-\frac{1}{2}} \otimes \mathfrak{V}_{\frac{1}{2}}, \quad \mathfrak{g}_1 \cong \wedge^2 \mathfrak{V}_{\frac{1}{2}}.$$

In particular,  $\mathfrak{g}_1$  and  $\mathfrak{g}_{-1}$  are dual to one another, and  $\mathfrak{g}_0$  is isomorphic to  $\mathfrak{gl}(m, \mathbb{C})$ , the Lie algebra of the general linear group  $\text{GL}(m, \mathbb{C})$  with standard representation  $\mathfrak{V}_{\frac{1}{2}}$ . We can then write

$$\phi_{ab} = \eta_{aA} \eta_{bB} \phi^{AB} + 2 \xi_{[a}{}^A \eta_{b]B} \phi_A{}^B + \xi_a{}^A \xi_b{}^B \phi_{AB} \in \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where  $\phi^{AB} = \phi^{[AB]} \in \wedge^2 \mathfrak{S}^{\frac{m-2}{4}}$ ,  $\phi_A{}^B \in \mathfrak{S}_{-\frac{m-2}{4}} \otimes \mathfrak{S}^{\frac{m-2}{4}}$ ,  $\phi_{AB} = \phi_{[AB]} \in \wedge^2 \mathfrak{S}_{-\frac{m-2}{4}}$ .

In this splitting, the parabolic Lie subalgebra  $\mathfrak{p}$  is given in terms of its Levy decomposition  $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ . By the commutation relation,  $\mathfrak{g}_1$  is nilpotent. Further, since  $\mathfrak{g}_0$  is reductive, there is a direct sum decomposition  $\mathfrak{g}_0 = \mathfrak{z}_0 \oplus \mathfrak{sl}_0$  where  $\mathfrak{z}_0$  is the one-dimensional center of  $\mathfrak{g}_0$ , and  $\mathfrak{sl}_0$  is the simple part of  $\mathfrak{g}_0$ , which is isomorphic to  $\mathfrak{sl}(m, \mathbb{C})$ , the Lie algebra of the special linear group  $\text{SL}(m, \mathbb{C})$ . The center  $\mathfrak{z}_0$  can be seen to be spanned by the element

$$E_{ab} := -\xi_{[a}{}^A \eta_{b]A} = -\xi_a{}^A \eta_{bA} + \frac{1}{2} g_{ab}, \quad (2.23)$$

with respect to which any  $\phi_{ab} \in \mathfrak{sl}_0$  is tracefree, i.e.  $E^{ab} \phi_{ab} = 0$ . More generally, any  $\phi_{ab} \in \mathfrak{g}_0$  admits the spinorial decomposition

$$\phi_{ab} = \phi \omega_{ab} + 2 \xi_{[a}{}^A \eta_{b]B} \phi_A{}^B \in \mathfrak{z}_0 \oplus \mathfrak{sl}_0,$$

where  $\phi \in \mathbb{C}$  and  $\phi_A^B \in \mathfrak{S}_{-\frac{m-2}{4}} \otimes \mathfrak{S}_{\frac{m-2}{4}}$  is tracefree in the sense that<sup>4</sup>  $\phi_A^B I_B^A = 0$ . Here, we have defined, for convenience,<sup>5</sup>

$$\omega_{ab} := -2E_{ab}, \quad (2.24)$$

so that  $\omega_a^c \omega_c^b = g_a^b$ .

This distinguished element  $E_{ab}$  has the property

$$\xi^{bA} E_b^a = \frac{1}{2} \xi^{aA}, \quad \eta^b{}_A E_b^a = -\frac{1}{2} \eta^a{}_A,$$

i.e.  $E_{ab}$  has eigenvalues  $\pm \frac{1}{2}$  on  $\mathfrak{V}_{\pm \frac{1}{2}}$ . The action of  $E_{ab}$  extends by derivation to any tensor product of  $\mathfrak{V}$  and  $\mathfrak{V}^*$ , and in particular  $E_{ab}$  has eigenvalues  $i$  on  $\mathfrak{g}_i$  for  $i = -1, 0, 1$ . Now, the image of  $E_{ab}$  in the Clifford algebra  $\mathcal{Cl}(\mathfrak{V}, g)$  restricted to  $\text{End}(\mathfrak{S}^+)$  is  $E_{B'}^{A'} := -\frac{1}{4} E_{ab} \gamma^{ab}{}_{B'}^{A'}$ , and has eigenvalues  $\frac{m}{4}$  on  $\mathfrak{S}_{\frac{m}{4}}$  since

$$E_{B'}^{A'} \xi^{B'} = \frac{m}{4} \xi^{A'},$$

and similarly for the action of  $E_{ab}$  on  $\mathfrak{S}^-$  and their duals. For these reasons,  $E_{ab}$  is referred to as *the grading element* of  $\mathfrak{g}$ .

Finally, we note that  $\mathfrak{g}_{\pm 1}^0 \cong \mathfrak{g}_{\pm 1}$ ,  $\mathfrak{g}_0^0 \cong \mathfrak{z}_0$  and  $\mathfrak{g}_0^1 \cong \mathfrak{sl}_0$  as *vector spaces*: as pointed out earlier each  $\mathfrak{g}_i^j \subset \mathfrak{g}^i / \mathfrak{g}^{i+1}$  are  $\mathfrak{p}$ -modules, but each of  $\mathfrak{g}_{\pm 1}$ ,  $\mathfrak{sl}_0$ , and  $\mathfrak{z}_0$  are irreducible  $\mathfrak{g}_0$ -modules. Since  $[\mathfrak{g}^1, \mathfrak{g}^i] \subset \mathfrak{g}^{i+1}$ , it is clear that  $\mathfrak{g}^1$  acts trivially on  $\text{gr}(\mathfrak{g})$  and in particular on each  $\mathfrak{g}_i^j$ , but the action of  $\mathfrak{g}_1$  on the  $\mathfrak{g}_0$ -modules is in general non-trivial. We can encode the additional information given by this action in terms of a ‘directed’ graph

$$\begin{array}{ccc} & \mathfrak{g}_0^0 & \\ \mathfrak{g}_1^0 \swarrow & & \searrow \mathfrak{g}_{-1}^0 \\ & \mathfrak{g}_0^1 & \end{array} \quad (2.25)$$

where the arrows are defined as

$$\mathfrak{g}_i^j \longrightarrow \mathfrak{g}_{i-1}^k \iff \check{\mathfrak{g}}_i^j \subset \mathfrak{g}_1 \cdot \check{\mathfrak{g}}_{i-1}^k, \quad (2.26)$$

and  $\check{\mathfrak{g}}_i^j$  is the irreducible  $\mathfrak{g}_0$ -module corresponding to  $\mathfrak{g}_i^j$ . In other words, the graph (2.25) encodes the irreducibility of the  $\mathfrak{p}$ -modules  $\mathfrak{g}_i^j$  together with the orbit of the nilpotent algebra  $\mathfrak{g}_1$  on the corresponding  $\mathfrak{g}_0$ -modules. Equivalently, and dually, one draws an arrow  $\mathfrak{g}_i^j \longrightarrow \mathfrak{g}_{i-1}^k$  for some  $i, j, k$  whenever the following statement is true:

$$\text{if } \phi \in \mathfrak{g} \text{ satisfies } {}^{\mathfrak{g}}\Pi_i^j(\phi) = 0 \text{ then } {}^{\mathfrak{g}}\Pi_{i-1}^k(\phi) = 0. \quad (2.27)$$

In the case at hand, this can easily be verified by means of the identity

$${}^{\mathfrak{g}}\Pi_{-1}^0(\phi)^{AB} = -\frac{1}{4} {}^{\mathfrak{g}}\Pi_0^0(\phi)^{C'} \gamma_{C'}^c \xi_c^A \xi_c^B = {}^{\mathfrak{g}}\Pi_0^1(\phi)^A \xi_c^{cB},$$

for any  $\phi_{ab} \in \mathfrak{g}$ . While this construction seems rather trivial in the case of  $\mathfrak{g}$ , we shall apply it to more complicated  $\mathfrak{g}$ -modules in the course of this article.

<sup>4</sup>Recall that  $I_A^B$  is the identity on  $\mathfrak{S}_{\frac{m-2}{4}}$  as defined by (2.9).

<sup>5</sup>Another convenient choice would be  $\omega_{ab} := -2iE_{ab}$ .

**Remark 2.14** By Lemma 2.11, an element of  $\mathfrak{g}^0$  takes the form  $\phi_{ab} = \xi_{[a}{}^A \phi_{Ab]}$  for some  $\phi_{Ab}$ . If it lies in  $\mathfrak{g}_0^1$ , then its spinor representative can be seen to satisfy  $\xi^{bA} \phi_{Ab} = 0$  in addition. The degeneracy to  $\mathfrak{g}_1^0$  is identical to that given in Remark 2.12.

Complementarily, an element of  $\mathfrak{g}_0^0$  takes the form  $\phi_{ab} = \xi_{ab}{}^{B'} \phi_{B'}$  for some  $\phi_{B'} \in \mathfrak{S}^{-\frac{m}{4}} / \mathfrak{S}^{-\frac{m-4}{4}}$ , and its degeneracy to  $\mathfrak{g}_1^0$  is mirrored by the fact that  $\phi_{B'}$  degenerates to a non-trivial element of  $\mathfrak{S}^{-\frac{m-4}{4}} / \mathfrak{S}^{-\frac{m-8}{4}}$ . In particular,  $\xi^{B'} \phi_{B'} = 0$ , so that  ${}^g\Pi_1(\phi) = 0$ . By Lemma 2.11, we can then write  $\phi_{ab} = \xi_a{}^C \xi_b{}^D \phi_{CD}$  for some  $\phi_{AB} = \phi_{[AB]} \in \wedge^2 \left( \mathfrak{S}^{-\frac{m-2}{4}} / \mathfrak{S}^{-\frac{m-6}{4}} \right)$ .

**Parabolic Lie subgroups** The complex special orthogonal group  $G := \mathrm{SO}(2m, \mathbb{C})$  has Lie algebra  $\mathfrak{g} = \mathfrak{so}(2m, \mathbb{C})$  as described above, and the subgroup  $P$  of  $G$  with Lie algebra  $\mathfrak{p}$  obtained by exponentiation is known as a *parabolic Lie subgroup* of  $G$ . It admits a Levy decomposition  $P = G_0 \ltimes P_+$ , where  $G_0$  is isomorphic to  $\mathrm{GL}(m, \mathbb{C})$  and  $P_+$  is the abelian Lie group with Lie algebra  $\mathfrak{g}_1$  [CS09, TC12]. The  $\mathfrak{p}$ -invariant filtrations and associated graded vector spaces considered in this article are also  $P$ -invariant and can be regarded as finite representations of  $P$ .

By definition, the spaces of projective positive and negative pure spinors can be identified with the null (or isotropic) Grassmannian  $\mathrm{Gr}_m^+(\mathfrak{V}, g)$  and  $\mathrm{Gr}_m^-(\mathfrak{V}, g)$  of  $\alpha$ -planes and  $\beta$ -planes in  $\mathfrak{V}$  respectively.<sup>6</sup> These are compact complex subvarieties of  $\mathbb{P}\mathfrak{S}^+$  and  $\mathbb{P}\mathfrak{S}^-$  respectively defined by the purity conditions (2.8). Now  $\mathrm{Gr}_m^+(\mathfrak{V}, g)$  and  $\mathrm{Gr}_m^-(\mathfrak{V}, g)$  are both  $\mathrm{SO}(2m, \mathbb{C})$ -orbits, each corresponding to a connected component of the complex orthogonal group  $\mathrm{O}(2m, \mathbb{C})$ . Since the parabolic  $P$  stabilises a point in  $\mathrm{Gr}_m^+(\mathfrak{V}, g)$ , we can thus identify  $\mathrm{Gr}_m^+(\mathfrak{V}, g)$  with the  $\frac{1}{2}m(m-1)$ -dimensional homogeneous space  $G/P$ , and similarly for  $\mathrm{Gr}_m^-(\mathfrak{V}, g)$ . When  $m = 1, 2, 3$ ,  $\mathrm{Gr}_m^+(\mathfrak{V}, g)$  and  $\mathrm{Gr}_m^-(\mathfrak{V}, g)$  are each isomorphic to the complex projective space  $\mathbb{CP}^{\frac{1}{2}m(m-1)}$ . More details can be found in e.g. [FH91, HS92].

**Remark 2.15** By Proposition 2.9, each (non-projective) positive pure spinor defines an  $m$ -dimensional vector subspace of (unprojective) negative pure spinors. Thus, to each point in  $\mathrm{Gr}_m^\pm(\mathfrak{V}, g)$  corresponds a  $(m-1)$ -dimensional complex projective space  $\mathbb{CP}^{m-1}$  in  $\mathrm{Gr}_m^\mp(\mathfrak{V}, g)$ . We note the following special cases:

- When  $m = 2$ ,  $\mathrm{Gr}_2^+(\mathfrak{V}, g)$  and  $\mathrm{Gr}_2^-(\mathfrak{V}, g)$  are each isomorphic to  $\mathbb{CP}^1$ , and the above remark is trivial.
- When  $m = 3$ ,  $\mathrm{Gr}_3^+(\mathfrak{V}, g)$  and  $\mathrm{Gr}_3^-(\mathfrak{V}, g)$  are dual to each other and each isomorphic to  $\mathbb{CP}^3$ . But points and  $\mathbb{CP}^2$ 's are dual to each other in  $\mathbb{CP}^3$ .
- When  $m = 4$ , there is a triality between  $\mathrm{Gr}_4^+(\mathfrak{V}, g)$ ,  $\mathrm{Gr}_4^-(\mathfrak{V}, g)$  and  $\mathcal{Q} := \{g(X, X) = 0 : [X] \in \mathbb{P}\mathfrak{V}\}$ , each of which are six-dimensional projective quadrics in  $\mathbb{CP}^7$ . A point in any of these spaces determines a  $\mathbb{CP}^3$  in the other two.

## 2.4 Filtrations and grading on $\mathfrak{S}$

The above calculus can be generalised to the whole exterior algebra and spinor spaces. Given a (projective) (positive) pure spinor  $\xi^{A'}$  we can define maps of vector spaces

$$\xi_{a_1 \dots a_{2k+1}}{}^A := \xi^{B'} \gamma_{a_1 \dots a_{2k+1} B'}{}^A : \wedge^{2k+1} \mathfrak{V} \rightarrow \mathfrak{S}^-, \quad \xi_{a_2 \dots a_{2k}}{}^{A'} := \xi^{B'} \gamma_{a_1 \dots a_{2k} B'}{}^{A'} : \wedge^{2k} \mathfrak{V} \rightarrow \mathfrak{S}^+.$$

By Hodge star duality  $\wedge^k \mathfrak{V} \cong \wedge^{2m-k} \mathfrak{V}$ , only forms of degree from 0 to  $m$  need to be considered with the understanding that  $\xi^{A'} : \wedge^0 \mathfrak{V} \cong \mathbb{C} \rightarrow \mathfrak{S}^+$ . An argument identical to that leading to (2.18) and (2.20) tells

<sup>6</sup>In this context, one usually regards  $\alpha$ -planes and  $\beta$ -planes as  $(m-1)$ -dimensional linear subspaces of the projectivisation  $\mathbb{P}\mathfrak{V}$  of  $\mathfrak{V}$ .

us that  $\xi^{A'}$  induces filtrations

$$\left. \begin{aligned} \mathfrak{S}_{\frac{m}{4}} &\subset \mathfrak{S}_{\frac{m-4}{4}} \subset \dots \subset \mathfrak{S}_{-\frac{m-4}{4}} \subset \mathfrak{S}_{-\frac{m}{4}} = \mathfrak{S}^+ \\ \mathfrak{S}_{\frac{m-2}{4}} &\subset \mathfrak{S}_{\frac{m-6}{4}} \subset \dots \subset \mathfrak{S}_{-\frac{m-6}{4}} \subset \mathfrak{S}_{-\frac{m-2}{4}} = \mathfrak{S}^- \end{aligned} \right\} \quad \text{when } m \text{ is even,}$$

$$\left. \begin{aligned} \mathfrak{S}_{\frac{m}{4}} &\subset \mathfrak{S}_{\frac{m-4}{4}} \subset \dots \subset \mathfrak{S}_{-\frac{m-6}{4}} \subset \mathfrak{S}_{-\frac{m-2}{4}} = \mathfrak{S}^+ \\ \mathfrak{S}_{\frac{m-2}{4}} &\subset \mathfrak{S}_{\frac{m-6}{4}} \subset \dots \subset \mathfrak{S}_{-\frac{m-4}{4}} \subset \mathfrak{S}_{-\frac{m}{4}} = \mathfrak{S}^- \end{aligned} \right\} \quad \text{when } m \text{ is odd,}$$

where we have defined

$$\mathfrak{S}_{\frac{m-4k-2}{4}} := \text{im } \xi_{a_1 \dots a_{2k+1}}^A : \wedge^{2k+1} \mathfrak{V} \rightarrow \mathfrak{S}^-, \quad \mathfrak{S}_{\frac{m-4k}{4}} := \text{im } \xi_{a_1 \dots a_{2k}}^{A'} : \wedge^{2k} \mathfrak{V} \rightarrow \mathfrak{S}^+.$$

Using the isomorphisms (2.3), the above filtrations are also filtrations on the dual spinor spaces  $(\mathfrak{S}^\pm)^*$ , where each of the vector subspaces can be identified with the kernels

$$\mathfrak{S}_{\frac{m-4k-4}{4}} = \ker \xi_{a_1 \dots a_{2k}}^{B'} : \wedge^{2k} \mathfrak{V}^* \leftarrow (\mathfrak{S}^*)^+, \quad \mathfrak{S}_{\frac{m-4k-6}{4}} = \ker \xi_{a_1 \dots a_{2k+1}}^B : \wedge^{2k+1} \mathfrak{V}^* \leftarrow (\mathfrak{S}^*)^-.$$

A choice of a spinor  $\eta_{A'}$  dual to  $\xi^{A'}$  fixes vector subspaces  $\mathfrak{S}_i \subset \mathfrak{S}^i$  such that  $\mathfrak{S}^i = \mathfrak{S}_i \oplus \mathfrak{S}^{i+1}$  and thus induces gradings

$$\left. \begin{aligned} \mathfrak{S}_{\frac{m}{4}} \oplus \mathfrak{S}_{\frac{m-4}{4}} \oplus \dots \oplus \mathfrak{S}_{-\frac{m-4}{4}} \oplus \mathfrak{S}_{-\frac{m}{4}} &= \mathfrak{S}^+ \\ \mathfrak{S}_{\frac{m-2}{4}} \oplus \mathfrak{S}_{\frac{m-6}{4}} \oplus \dots \oplus \mathfrak{S}_{-\frac{m-6}{4}} \oplus \mathfrak{S}_{-\frac{m-2}{4}} &= \mathfrak{S}^- \end{aligned} \right\} \quad \text{when } m \text{ is even,}$$

$$\left. \begin{aligned} \mathfrak{S}_{\frac{m}{4}} \oplus \mathfrak{S}_{\frac{m-4}{4}} \oplus \dots \oplus \mathfrak{S}_{-\frac{m-6}{4}} \oplus \mathfrak{S}_{-\frac{m-2}{4}} &= \mathfrak{S}^+ \\ \mathfrak{S}_{\frac{m-2}{4}} \oplus \mathfrak{S}_{\frac{m-6}{4}} \oplus \dots \oplus \mathfrak{S}_{-\frac{m-4}{4}} \oplus \mathfrak{S}_{-\frac{m}{4}} &= \mathfrak{S}^- \end{aligned} \right\} \quad \text{when } m \text{ is odd.}$$

Further, the grading element  $E$  has eigenvalues  $\frac{2i-m}{4}$  on  $\mathfrak{S}_{\frac{2i-m}{4}}$ .

**Remark 2.16** If one identifies the spinor representation with  $\wedge^\bullet \mathfrak{N}$  where  $\mathfrak{N}$  is the totally null  $m$ -dimensional annihilating  $\xi^{A'}$  as in Remark 2.1, then we have the identification  $\mathfrak{S}_{\frac{m-2i}{4}} \cong \wedge^{m-i} \mathfrak{N}$ .

Finally generalising (2.6) and (2.21), one has isomorphisms

$$\begin{aligned} \wedge^k \mathfrak{V}^{\frac{1}{2}} &\cong \mathfrak{S}_{\frac{m}{4}} \otimes \left( \mathfrak{S}_{-\frac{m-2k}{4}} / \mathfrak{S}_{-\frac{m-2k-4}{4}} \right), & k = 0, \dots, m-1, \\ \wedge^m \mathfrak{V}^{\frac{1}{2}} &\cong \mathfrak{S}_{\frac{m}{4}} \otimes \mathfrak{S}_{\frac{m}{4}}, \end{aligned}$$

the latter being the purity condition of Proposition 2.8.

## 2.5 Real pure spinors

One can also consider a  $2m$ -dimensional real vector space  $\mathfrak{V}$  equipped with a definite or indefinite non-degenerate symmetric bilinear form  $g_{ab}$ . In general, the spinor representations of  $(\mathfrak{V}, g)$  are complex vector spaces equipped with a real or quaternionic structure. The geometric object of interest here is not a single pure spinor, but a *complex conjugate pair* of pure spinors, which define a conjugate pair of complex totally null  $m$ -planes in the complexification of  $\mathfrak{V}$ . As already briefly explained in [TC12], this means that the analogue of the present work in this real setting is based on classifications of tensors invariant under the stabiliser of a complex conjugate pair of pure spinors.

For instance, when  $g_{ab}$  is positive definite, this stabiliser is the unitary group  $U(m)$ , which being reductive, leads to  $U(m)$ -invariant direct sum decompositions of  $SO(2m)$ -modules [GH80, TV81, FFS94]. When  $g_{ab}$  has Lorentzian signature: the stabiliser of the complex conjugate pair of pure spinors is now a subgroup of the  $\text{Sim}(n-2)$  group. The story is more complicated, and the classification of curvature tensors in this context is treated in a separate article [TCb].

However, when  $g_{ab}$  has signature  $(m, m)$ , there exist *real* pure spinors and *real* totally null  $m$ -planes in  $\mathfrak{V}$ . The algebraic setup of the previous sections carries over to this real setting with no major change. The complex Lie algebra  $\mathfrak{so}(2m, \mathbb{C})$  is replaced by the real form  $\mathfrak{so}(m, m)$ . The parabolic Lie subalgebra stabilising a real pure spinor is a real form of the complex parabolic  $\mathfrak{p}$ , and is also described in terms of a 1-grading on  $\mathfrak{so}(m, m)$ . The story is similar at the Lie group level, where  $\mathrm{SO}(2m, \mathbb{C})$  is replaced by the connected identity component of the real Lie group  $\mathrm{SO}(m, m)$ . The subsequent classification of curvature tensors and intrinsic torsion, together with their relation to spinorial differential equations, as described in the rest of this article, can also be translated into this real case with no important issue.

### 3 Algebraic classifications of curvature tensors

It is time to apply the ideas introduced in section 2 to the classification of curvature tensors in terms of a given (projective) (positive) pure spinor  $\xi^{A'}$ . However, while one could use ad-hoc methods to achieve this purpose, we shall take a short-cut in this endeavour by appealing to Lie representation theory. As before  $\mathfrak{g} = \mathfrak{so}(2m, \mathbb{C})$  and  $\mathfrak{p}$  is the parabolic Lie subalgebra stabilising the pure spinor  $\xi^{A'}$ . We shall assume  $m > 2$  for definiteness, the special case  $m = 2$  being briefly covered in appendix A.1 To carry out the classification, we follow the following recipe based on [ČS09]:

1. Starting with a finite irreducible  $\mathfrak{g}$ -module  $\mathfrak{R}$ ,
2. we obtain a filtration

$$\{0\} =: \mathfrak{R}^{k+1} \subset \mathfrak{R}^k \subset \mathfrak{R}^{k-1} \subset \dots \subset \mathfrak{R}^{-k+1} \subset \mathfrak{R}^{-k} := \mathfrak{R},$$

for some  $k$ , of  $\mathfrak{p}$ -modules,

3. with associated graded  $\mathfrak{p}$ -module  $\mathrm{gr}(\mathfrak{R}) = \bigoplus \mathrm{gr}_i(\mathfrak{R})$ , where  $\mathrm{gr}_i(\mathfrak{R}) := \mathfrak{R}^i / \mathfrak{R}^{i+1}$ , on which the grading element  $E$  acts diagonalisably, with eigenvalues  $i$ ;
4. each  $\mathrm{gr}_i(\mathfrak{R})$  splits as a direct sum of irreducible  $\mathfrak{p}$ -modules

$$\mathrm{gr}_i(\mathfrak{R}) = \mathfrak{R}_i^0 \oplus \mathfrak{R}_i^1 \oplus \dots \oplus \mathfrak{R}_i^\ell$$

for some  $\ell$  depending on  $i$ , and each  $\mathfrak{R}_i^j$  isomorphic to an irreducible  $\mathfrak{g}_0$ -module  $\check{\mathfrak{R}}_i^j$ ;

5. we let the nilpotent part  $\mathfrak{g}_1$  of  $\mathfrak{p}$  act on each  $\check{\mathfrak{R}}_i^j$ , and draw an arrow  $\mathfrak{R}_i^j \rightarrow \mathfrak{R}_{i-1}^k$  for some  $i, j, k$ , whenever  $\check{\mathfrak{R}}_i^j \subset \mathfrak{g}_1 \cdot \check{\mathfrak{R}}_{i-1}^k$ . This completes the construction of the ‘Penrose diagram’ of  $\mathfrak{R}$  with respect to  $\mathfrak{p}$ .

An explicit description of the irreducible  $\mathfrak{p}$ -modules occurring in point 4 of this algorithm can be found in appendix C.

So much for the representation theoretic part of the classification. The remaining part consists in describing the decomposition of the Weyl tensor in terms of kernels of maps analogous to (2.15) and (2.22), as we have done in the description of  $\mathfrak{g}$ . Concretely, for each relevant  $i, j$ , we shall construct a map  ${}^{\mathfrak{R}}\Pi_i^j$ , which on restriction to  $\mathrm{gr}_i(\mathfrak{R})$  coincides with the projection to the irreducible summand  $\mathfrak{R}_i^j$ . From the irreducibility of  $\mathfrak{R}_i^j$ , each  ${}^{\mathfrak{R}}\Pi_i^j$  must be ‘saturated’ with symmetries in the sense of [PR84]. This also yields a dual way of drawing an arrow  $\mathfrak{R}_i^j \rightarrow \mathfrak{R}_{i-1}^k$ , simply by checking the veracity of the statement

$$\text{if } R \in \mathfrak{R} \text{ satisfies } {}^{\mathfrak{R}}\Pi_i^j(R) = 0 \text{ then } {}^{\mathfrak{R}}\Pi_{i-1}^k(R) = 0.$$

In addition, we shall give spinorial formulae for elements of each of the irreducible  $\mathfrak{g}_0$ -modules  $\check{\mathfrak{R}}_i^j$ . These can be obtained by reading off the dominant weight of the representations as explained in appendix C.

**Remark 3.1 (Notation)** The notation of section 2 will be used throughout. In particular, we shall make use of the maps  $\xi^{aA}$  so that upstairs and downstairs upper case unprimed Roman indices will refer to the  $\mathfrak{p}$ -modules  $\mathfrak{S}^{\frac{m-2}{4}}$  and  $\mathfrak{S}^{-\frac{m-2}{4}}/\mathfrak{S}^{-\frac{m-6}{4}}$  respectively. If one chooses a spinor  $\eta_{A'}$  dual to  $\xi^{A'}$ , then these will refer to the  $\mathfrak{g}_0$ -modules  $\mathfrak{S}_{\frac{m-2}{4}}$  and  $\mathfrak{S}_{-\frac{m-2}{4}}$  respectively, so that we shall write

$$\sigma_{A\dots C}{}^{D\dots F} \in \mathfrak{S}_{-\frac{m-2}{4}} \otimes \dots \otimes \mathfrak{S}_{-\frac{m-2}{4}} \otimes \mathfrak{S}^{\frac{m-2}{4}} \otimes \dots \otimes \mathfrak{S}^{\frac{m-2}{4}},$$

and such a spinorial will be referred as *(totally) tracefree* if the contraction of any pair of indices with the identity element  $I_A^B$  defined by (2.9) vanishes, e.g.

$$\sigma_A{}^B I_B^A = 0.$$

Finally, we recall the definition (2.24) of the 2-form  $\omega_{ab} := 2\xi_{[a}{}^A \eta_{b]A}$ .

### 3.1 The tracefree Ricci tensor

We start with the space  $\mathfrak{F} := \odot^2 \mathfrak{V}$  of tracefree symmetric 2-tensors

$$\mathfrak{F} := \{\Phi_{ab} \in \otimes^2 \mathfrak{V}^* : \Phi_{ab} = \Phi_{(ab)}, \Phi_c{}^c = 0\}.$$

For  $\Phi_{ab} \in \mathfrak{F}$ , we define

$$\mathfrak{F}\Pi_{-1}(\Phi) := \xi^{aA} \xi^{bB} \Phi_{ab}, \quad \mathfrak{F}\Pi_0(\Phi) := \xi^{aA} \Phi_{ab},$$

and vector subspaces  $\mathfrak{F}^i := \{\Phi_{ab} \in \mathfrak{F} : \mathfrak{F}\Pi_{i-1}(\Phi) = 0\}$  for  $i = 0, 1$ , so that  $\mathfrak{S}$  acquires a  $\mathfrak{p}$ -invariant filtration

$$\{0\} =: \mathfrak{F}^2 \subset \mathfrak{F}^1 \subset \mathfrak{F}^0 \subset \mathfrak{F}^{-1} := \mathfrak{F}.$$

There is an analogue of Lemmata 2.6 and 2.11 for elements of  $\text{gr}(\mathfrak{F})$ , which we shall omit however. Instead, we give spinorial decomposition of the  $\mathfrak{g}_0$ -modules  $\mathfrak{F}_i$ , which are irreducible.

**Lemma 3.2** *Let  $\Phi_{ab} \in \mathfrak{F}$ . Then*

- $\Phi_{ab} \in \mathfrak{F}_0$  if and only if  $\Phi_{ab} = 2\xi_{(a}{}^A \eta_{b)B} \Phi_A{}^B$  for some tracefree  $\Phi_A{}^B$ ;
- $\Phi_{ab} \in \mathfrak{F}_1$  if and only if  $\Phi_{ab} = \xi_a{}^A \xi_b{}^B \Phi_{AB}$  for some  $\Phi_{AB} = \Phi_{(AB)}$ , and similarly for  $\mathfrak{F}_{-1} \cong (\mathfrak{F}_1)^*$  by substituting  $\xi^{A'}$  for  $\eta_{A'}$ , and changing the index structure appropriately.

### 3.2 The Cotton-York tensor

**Filtration** For  $m > 2$ , let  $\mathfrak{A}$  denote the space of tensors with Cotton-York symmetries, i.e.

$$\mathfrak{A} := \{A_{abc} \in \otimes^3 \mathfrak{V} : A_{abc} = A_{a[bc]}, A_{[abc]} = 0, A^a{}_{ac} = 0\},$$

and for  $A_{abc} \in \mathfrak{A}$  define maps

$$\mathfrak{A}\Pi_{-\frac{3}{2}}(A) := \xi^{aA} \xi^{bB} \xi^{cC} A_{abc}, \quad \mathfrak{A}\Pi_{-\frac{1}{2}}(A) := \xi^{aA} \xi^{bB} A_{abc} \quad \mathfrak{A}\Pi_{\frac{1}{2}}(A) := A_{abc} \xi^{cC}. \quad (3.1)$$

Then the projective pure spinor  $\xi^{A'}$  induces a filtration

$$\{0\} =: \mathfrak{A}^{\frac{5}{2}} \subset \mathfrak{A}^{\frac{3}{2}} \subset \mathfrak{A}^{\frac{1}{2}} \subset \mathfrak{A}^{-\frac{1}{2}} \subset \mathfrak{A}^{-\frac{3}{2}} = \mathfrak{A},$$

of  $\mathfrak{p}$ -modules on  $\mathfrak{A}$ , where  $\mathfrak{A}^i = \{A_{abc} \in \mathfrak{A} : \mathfrak{A}\Pi_{i-1}(A) = 0\}$  for  $i = \pm\frac{1}{2}, \frac{3}{2}$ .



**Associated graded module** As before, each summand  $\text{gr}_i(\mathfrak{A}) := \mathfrak{A}^i / \mathfrak{A}^{i-1}$  of the associated graded  $\mathfrak{p}$ -module  $\text{gr}(\mathfrak{A})$  is a completely reducible  $\mathfrak{p}$ -module. For  $A_{abc} \in \mathfrak{A}$ , one can define maps

$$\begin{aligned}
\mathfrak{A}\Pi_{-\frac{3}{2}}^0(A) &:= \xi^{aA} \xi^{bB} \xi^{cC} A_{abc}, \\
\mathfrak{A}\Pi_{-\frac{1}{2}}^0(A) &:= \xi^{aA} A_{abc} \xi^{bcC'}, \\
\mathfrak{A}\Pi_{-\frac{1}{2}}^1(A) &:= \xi^{bB} \xi^{cC} A_{abc} + \frac{1}{n-2} \gamma_{aD'}^{[B} \xi^{dC]} \xi^{bcD'} A_{dbc}, \\
\mathfrak{A}\Pi_{-\frac{1}{2}}^2(A) &:= \xi^{a(A} \xi^{bB)} A_{abc} + \frac{3}{2(n+2)} \gamma_{cD'}^{(A} \xi^{dB)} \xi^{baD'} A_{dba}, \\
\mathfrak{A}\Pi_{\frac{1}{2}}^0(A) &:= A_{abc} \xi^{bcA'}, \\
\mathfrak{A}\Pi_{\frac{1}{2}}^1(A) &:= \xi^{cA} A_{cab} - \frac{1}{n-2} \gamma_{[aD'}^A A_{b]cd} \xi^{cdD'}, \\
\mathfrak{A}\Pi_{\frac{1}{2}}^2(A) &:= A_{(ab)c} \xi^{cA} - \frac{3}{2(n+2)} \gamma_{(aD'}^A A_{b)cd} \xi^{cdD'}.
\end{aligned} \tag{3.2}$$

These can be seen to be ‘saturated’ with symmetries, and thus must refer to irreducible  $\mathfrak{p}$ -modules of the associated graded vector space  $\text{gr}(\mathfrak{A})$ . In fact, one has

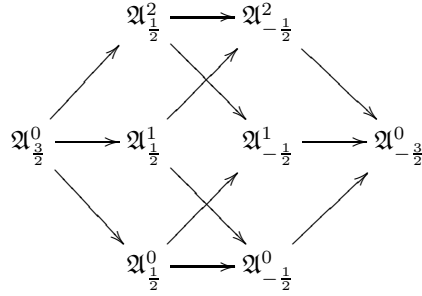
**Proposition 3.3** *For each  $i$ ,  $\text{gr}_i(\mathfrak{A}) := \mathfrak{A}^i / \mathfrak{A}^{i-1}$  splits into a direct sum of irreducible  $\mathfrak{p}$ -modules*

$$\text{gr}_{\pm\frac{3}{2}}(\mathfrak{A}) \cong \mathfrak{A}_{\pm\frac{3}{2}}^0, \quad \text{gr}_{\pm\frac{1}{2}}(\mathfrak{A}) \cong \mathfrak{A}_{\pm\frac{1}{2}}^0 \oplus \mathfrak{A}_{\pm\frac{1}{2}}^1 \oplus \mathfrak{A}_{\pm\frac{1}{2}}^2,$$

where

$$\mathfrak{A}_i^j = \{A_{abc} \in \mathfrak{A}^i : \mathfrak{A}\Pi_i^k(A) = 0, \text{ for all } k \neq j\} / \mathfrak{A}^{i+1}.$$

Further the structure of  $\text{gr}(\mathfrak{A})$  can be expressed by means of the graph



where arrows are defined according to the relation

$$\mathfrak{A}_i^j \longrightarrow \mathfrak{A}_{i-1}^k \quad \Longleftrightarrow \quad \check{\mathfrak{A}}_i^j \subset \mathfrak{g}_1 \cdot \check{\mathfrak{A}}_{i-1}^k,$$

where  $\check{\mathfrak{A}}_i^j$  is the irreducible  $\mathfrak{g}_0$ -module corresponding to  $\mathfrak{A}_i^j$ .

## Grading

**Lemma 3.4** *Let  $A_{abc} \in \mathfrak{A}$ . Then*

- $A_{abc} \in \check{\mathfrak{A}}_{\frac{1}{2}}^0$  if and only if

$$A_{abc} = A_a \omega_{bc} - A_{[b} \omega_{c]a} + \frac{3}{n-1} g_{a[b} \omega_{c]d} A^d.$$

for some  $A_c = \xi_c^C A_C$ ;

- $A_{abc} \in \check{\mathfrak{A}}_{\frac{1}{2}}^1$  if and only if

$$A_{abc} = \xi_b^A \xi_c^B \eta_{aC} A_{AB}^C - \xi_a^A \xi_{[b}^B \eta_{c]C} A_{AB}^C$$

for some tracefree  $A_{AB}^C = A_{[AB]}^C$ ;

- $A_{abc} \in \check{\mathfrak{A}}_{\frac{1}{2}}^2$  if and only if  $A_{abc} = 2\xi_a^A \xi_{[b}^B \eta_{c]C} A_{AB}^C$  for some tracefree  $A_{AB}^C = A_{(AB)}^C$ ;
- $A_{abc} \in \check{\mathfrak{A}}_{\frac{1}{2}}^0$  if and only if  $A_{abc} = \xi_a^A \xi_b^B \xi_c^C A_{ABC}$  for some  $A_{ABC} = A_{A[BC]}$  satisfying  $A_{[ABC]} = 0$ .

Since  $(\check{\mathfrak{A}}_i^j)^* \cong \check{\mathfrak{A}}_{-i}^j$ , spinorial formulae for elements of  $\check{\mathfrak{A}}_{-i}^j$  for  $i > 0$  can be obtained from those of  $\check{\mathfrak{A}}_i^j$  by simply interchanging  $\xi^{A'}$  and  $\eta_{A'}$  and making appropriate changes of index structures.

### 3.3 The Weyl tensor

**Filtration** For  $m > 2$ , let  $\mathfrak{C}$  denote the space of tensors with Weyl symmetries, i.e.

$$\mathfrak{C} := \{C_{abcd} \in \otimes^4 \mathfrak{V} : C_{abcd} = C_{[ab][cd]}, C_{[abc]d} = 0, C_{bad}^a = 0\},$$

and for  $C_{abcd} \in \mathfrak{C}$ , define

$$\begin{aligned} \mathfrak{C}\Pi_{-2}(C) &:= \xi^{aA} \xi^{bB} \xi^{cC} \xi^{dD} C_{abcd}, & \mathfrak{C}\Pi_{-1}(C) &:= \xi^{aA} \xi^{bB} \xi^{cC} C_{abcd}, \\ \mathfrak{C}\Pi_0(C) &:= \xi^{aA} \xi^{cC} C_{abcd}, & \mathfrak{C}\Pi_1(C) &:= \xi^{aA} C_{abcd}. \end{aligned} \quad (3.3)$$

Then the projective pure spinor  $\xi^{A'}$  induces a filtration

$$\{0\} =: \mathfrak{C}^3 \subset \mathfrak{C}^2 \subset \mathfrak{C}^1 \subset \mathfrak{C}^0 \subset \mathfrak{C}^{-1} \subset \mathfrak{C}^{-2} := \mathfrak{C}, \quad (3.4)$$

of  $\mathfrak{p}$ -modules on  $\mathfrak{C}$ , where  $\mathfrak{C}^i = \{C_{abcd} \in \mathfrak{C} : \mathfrak{C}\Pi_{i-1}(C) = 0\}$  for  $i = \pm 1, 0, 2$ . This is clear since  $\mathfrak{C}\Pi_i$  is related to  $\mathfrak{C}\Pi_{i-1}$  via contraction with  $\xi^{aB}$ .

As for the tracefree Ricci tensor, there is an analogue of Lemmata 2.6 and 2.11 for the Weyl tensor, which we shall omit. For instance,  $C_{abcd} \in \mathfrak{C}_{-1}$  if and only if

$$C_{abcd} = \xi_{[a}^A C_{Ab]cd} + \xi_{[c}^A C_{Ad]ab} + \frac{2}{n-2} \left( g_{[a|[c} C_{A d]| b]e} \xi^{Ae} + g_{[c|[a} C_{A b]| d]e} \xi^{Ae} \right).$$

for some  $C_{Abcd} = C_{Ab[cd]}$  satisfying  $C_{A[bcd]} = 0$  and  $C_{Abc}{}^b = 0$ .

**Associated graded module** Contracting  $\mathfrak{C}\Pi_{-1}$ ,  $\mathfrak{C}\Pi_0$  and  $\mathfrak{C}\Pi_1$  with  $\gamma$ -matrices clearly is not a vacuous operation, which suggests that not all the vector subspaces  $\mathfrak{C}^i/\mathfrak{C}^{i+1}$  are irreducible. By saturating the maps

(3.3) with symmetries, we obtain, for  $C_{abcd} \in \mathfrak{C}$ , the maps

$$\begin{aligned}
\mathfrak{C}\Pi_{-2}^0(C) &:= \xi^{aA} \xi^{bB} \xi^{cC} \xi^{dD} C_{abcd}, \\
\mathfrak{C}\Pi_{-1}^0(C) &:= \xi^{aA} \xi^{bB} \xi^{cdC'} C_{abcd}, \\
\mathfrak{C}\Pi_{-1}^1(C) &:= \xi^{aA} \xi^{bB} \xi^{cC} C_{abce} + \frac{1}{n+2} \left( \xi^{aA} \xi^{bB} \xi^{cdD'} C_{abcd} \gamma_{eD'}^C - \xi^{aC} \xi^{b[A} \xi^{cdD'} C_{abcd} \gamma_{eD'}^{B]} \right), \\
\mathfrak{C}\Pi_0^0(C) &:= \xi^{abB'} \xi^{cdD'} C_{abcd}, \\
\mathfrak{C}\Pi_0^1(C) &:= \xi^{abB'} \xi^{cD} C_{abcd} + \frac{1}{n} \xi^{abB'} \xi^{ceD'} C_{abce} \gamma_{dD'}^D, \\
\mathfrak{C}\Pi_0^2(C) &:= \xi^{aA} C_{a[bcd]d} \xi^{dD} + \frac{1}{n-4} \xi^{aeC'} C_{aed[b} \gamma_{c]C'}^{[A} \xi^{dD]} - \frac{1}{2(n-2)(n-4)} \xi^{aeC'} C_{aedf} \xi^{dfF'} \gamma_{[bC'}^A \gamma_{c]F'}^D, \\
\mathfrak{C}\Pi_0^3(C) &:= \xi^{aA} C_{a(bc)d} \xi^{dD} - \frac{3}{n+4} \xi^{aeC'} C_{aed(b} \gamma_{c)C'}^{(A} \xi^{dD)} \\
&\quad - \frac{3}{2(n+2)(n+4)} \xi^{aeC'} C_{aedf} \xi^{dfF'} \gamma_{(bC'}^A \gamma_{c)F'}^D, \\
\mathfrak{C}\Pi_1^0(C) &:= \xi^{abB'} C_{abcd}, \\
\mathfrak{C}\Pi_1^1(C) &:= \xi^{aB} C_{abcd} + \frac{1}{n+2} \left( \xi^{aeC'} \gamma_{bC'}^B C_{aec d} - \xi^{aeC'} C_{aeb[c} \gamma_{d]C'}^B \right),
\end{aligned} \tag{3.5}$$

with the proviso that  $\mathfrak{C}\Pi_0^2$  does not occur when  $m = 3$ .

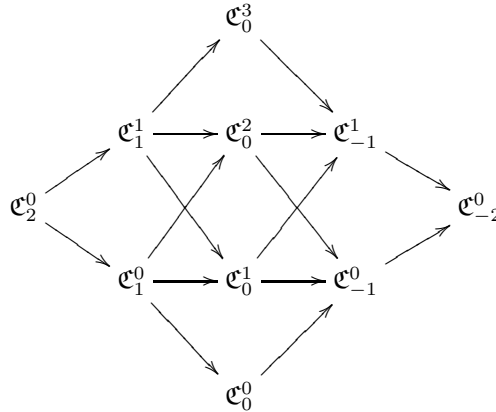
**Proposition 3.5** *Each summand  $\text{gr}_i(\mathfrak{C}) := \mathfrak{C}^i / \mathfrak{C}^{i+1}$  of the associated graded  $\mathfrak{p}$ -module  $\text{gr}(\mathfrak{C})$  splits as a direct sum of irreducible  $\mathfrak{p}$ -modules*

$$\text{gr}_{\pm 2}(\mathfrak{C}) \cong \mathfrak{C}_{\pm 2}^0, \quad \text{gr}_{\pm 1}(\mathfrak{C}) \cong \mathfrak{C}_{\pm 1}^0 \oplus \mathfrak{C}_{\pm 1}^1, \quad \text{gr}_0(\mathfrak{C}) \cong \mathfrak{C}_0^0 \oplus \mathfrak{C}_0^1 \oplus \mathfrak{C}_0^2 \oplus \mathfrak{C}_0^3,$$

where

$$\mathfrak{C}_i^j = \{C \in \mathfrak{C}^i : \mathfrak{C}\Pi_i^k(C) = 0, \text{ for all } k \neq j\} / \mathfrak{C}^{i+1}.$$

Further, the space of tensors with Weyl symmetries can be expressed by means of a  $\mathfrak{p}$ -invariant graph



with the proviso that  $\mathfrak{C}_0^2$  does not occur when  $m = 3$ . Here, arrows are defined according to the relation

$$\mathfrak{C}_i^j \longrightarrow \mathfrak{C}_{i-1}^k \quad \Longleftrightarrow \quad \check{\mathfrak{C}}_i^j \subset \mathfrak{g}_1 \cdot \check{\mathfrak{C}}_{i-1}^k,$$

where  $\check{\mathfrak{C}}_i^j$  is the irreducible  $\mathfrak{g}_0$ -module corresponding to  $\mathfrak{C}_i^j$ .

## Grading

**Lemma 3.6** *Let  $C_{abcd} \in \mathfrak{C}$ . Then*

- $C_{abcd} \in \check{\mathfrak{C}}_0^0$  if and only if

$$C_{abcd} = c \left( 2\omega_{ab}\omega_{cd} - 2\omega_{a[c}\omega_{d]b} + \frac{6}{n-1} \left( g_{a[c}g_{d]b} \right) \right),$$

for some complex  $c$ ;

- $C_{abcd} \in \check{\mathfrak{C}}_0^1$  if and only if

$$C_{abcd} = \omega_{ab}C_{cd} + C_{ab}\omega_{cd} - 2\omega_{[a|[c}C_{d]|b]} - \frac{6}{n-2} \left( g_{[a|[c}\omega_{d]}{}^e C_{|b]e} + g_{[c|[a}\omega_{b]}{}^e C_{|d]e} \right).$$

where  $C_{cd} := 2\xi_{[c}{}^C\eta_{d]D}C_C{}^D$  for some tracefree  $C_C{}^D$ ;

- when  $m > 3$ ,  $C_{abcd} \in \check{\mathfrak{C}}_0^2$  if and only if

$$C_{abcd} = \xi_a{}^A\xi_b{}^B\eta_{cC}\eta_{dD}C_{AB}{}^{CD} + \xi_c{}^A\xi_d{}^B\eta_{aC}\eta_{bD}C_{AB}{}^{CD} - 2\xi_{[a}{}^A\xi_{[c}{}^C\eta_{d]|D}\eta_{b]B}C_{AC}{}^{DB},$$

for some tracefree  $C_{AC}{}^{DB} = C_{[AC]}{}^{[DB]}$ ;

- $C_{abcd} \in \check{\mathfrak{C}}_0^3$  if and only if  $C_{abcd} = 4\xi_{[a}{}^A\xi_{[c}{}^C\eta_{d]|D}\eta_{b]B}C_{AC}{}^{DB}$  for some tracefree  $C_{AC}{}^{DB} = C_{(AC)}{}^{(DB)}$ ;

- $C_{abcd} \in \check{\mathfrak{C}}_1^0$  if and only if

$$C_{abcd} = \omega_{ab}C_{cd} + C_{ab}\omega_{cd} - 2\omega_{[a|[c}C_{d]|b]} - \frac{6}{n-2} \left( g_{[a|[c}\omega_{d]}{}^e C_{|b]e} + g_{[c|[a}\omega_{b]}{}^e C_{|d]e} \right).$$

where  $C_{ab} := \xi_a{}^A\xi_b{}^B C_{AB}$  for some  $C_{CD} = C_{[CD]}$ ;

- $C_{abcd} \in \check{\mathfrak{C}}_1^1$  if and only if

$$C_{abcd} = 2\xi_a{}^A\xi_b{}^B\xi_{[c}{}^C\eta_{d]D}C_{ABC}{}^D + 2\xi_c{}^A\xi_d{}^B\xi_{[a}{}^C\eta_{b]D}C_{ABC}{}^D,$$

for some tracefree  $C_{ABC}{}^D = C_{[AB]C}{}^D$  satisfying  $C_{[ABC]}{}^D = 0$ ;

- $C_{abcd} \in \check{\mathfrak{C}}_2^0$  if and only if  $C_{abcd} = \xi_a{}^A\xi_b{}^B\xi_c{}^C\xi_d{}^D C_{ABCD}$  for some  $C_{ABCD} = C_{[AB][CD]}$  satisfying  $C_{[ABC]D} = 0$ .

Since  $(\check{\mathfrak{C}}_i^j)^* \cong \check{\mathfrak{C}}_{-i}^j$ , spinorial formulae for elements of  $\check{\mathfrak{C}}_{-i}^j$  for  $i > 0$  can be obtained from those of  $\check{\mathfrak{C}}_i^j$  by simply interchanging  $\xi^{A'}$  and  $\eta_{A'}$  and making appropriate changes of index structures.

## 4 The $\mathfrak{p}$ -invariant decomposition of $\mathfrak{V} \otimes \mathfrak{g}/\mathfrak{p}$

In this section, we decompose the  $\mathfrak{p}$ -module  $\mathfrak{W} := \mathfrak{V} \otimes \mathfrak{g}/\mathfrak{p}$  into irreducibles. The geometrical meaning of  $\mathfrak{W}$  will become apparent in section 5. Again, we assume  $m > 2$ . We first note that the filtration (2.4) on  $\mathfrak{V}$  induces a filtration

$$\mathfrak{W}^{-\frac{1}{2}} \subset \mathfrak{W}^{-\frac{3}{2}}, \quad (4.1)$$

of  $\mathfrak{p}$ -modules on  $\mathfrak{W}$ , where

$$\mathfrak{W}^{-\frac{3}{2}} := \mathfrak{V}^{-\frac{1}{2}} \otimes (\mathfrak{g}^{-1}/\mathfrak{g}^0), \quad \mathfrak{W}^{-\frac{1}{2}} := \mathfrak{V}^{\frac{1}{2}} \otimes (\mathfrak{g}^{-1}/\mathfrak{g}^0).$$

It is convenient to view an element of  $\mathfrak{W}^{-\frac{3}{2}}$  as an element of  $\mathfrak{V}^{-\frac{1}{2}} \otimes \left( \wedge^2 \mathfrak{S}^{-\frac{1}{2}} \right)$ , i.e. of the form  $\Gamma_{abc} \xi^{bB} \xi^{cC}$  or  $\Gamma_{abc} \xi^{bcB'}$  (mod  $\alpha_a \xi^{B'}$ ) where  $\Gamma_{abc} = \Gamma_{a[bC]}$  lies in the  $\mathfrak{g}$ -module  $\mathfrak{V} \otimes \mathfrak{g}$ . This means that we can express  $\mathfrak{W}^{-\frac{1}{2}}$  as

$$\mathfrak{W}^{-\frac{1}{2}} = \{ \Gamma_{abc} \xi^{bB} \xi^{cC} \in \mathfrak{W}^{-\frac{3}{2}} : \Gamma_{abc} \xi^{aA} \xi^{bB} \xi^{cC} = 0 \}.$$

The inclusion of  $\mathfrak{p}$ -modules (4.1) can be refined in terms of irreducible  $\mathfrak{p}$ -modules in a way very similar to the classification of curvature tensors by considering its associated graded  $\mathfrak{p}$ -module

$$\text{gr}(\mathfrak{W}) = \text{gr}_{-\frac{1}{2}}(\mathfrak{W}) \oplus \text{gr}_{-\frac{3}{2}}(\mathfrak{W}) = \mathfrak{W}^{-\frac{1}{2}} \oplus \left( \mathfrak{W}^{-\frac{3}{2}} / \mathfrak{W}^{-\frac{1}{2}} \right).$$

The only difference here is that our starting point  $\mathfrak{W}$  is a  $\mathfrak{p}$ -module, and not an irreducible  $\mathfrak{g}$ -module. To make the analysis more tractable, we can work with the grading (2.10) so that we have linear isomorphisms

$$\text{gr}_{-\frac{3}{2}}(\mathfrak{W}) \cong \mathfrak{V}_{-\frac{1}{2}} \otimes \left( \wedge^2 \mathfrak{V}_{-\frac{1}{2}} \right), \quad \text{gr}_{-\frac{1}{2}}(\mathfrak{W}) \cong \mathfrak{V}_{\frac{1}{2}} \otimes \left( \wedge^2 \mathfrak{V}_{-\frac{1}{2}} \right),$$

where the RHS of each isomorphism is a  $\mathfrak{g}_0$ -module. Thus, there exist irreducible  $\mathfrak{p}$ -modules  $\mathfrak{W}_{-\frac{3}{2}}^0, \mathfrak{W}_{-\frac{3}{2}}^1, \mathfrak{W}_{-\frac{1}{2}}^0$  and  $\mathfrak{W}_{-\frac{1}{2}}^1$  linearly isomorphic to irreducible  $\mathfrak{g}_0$ -modules  $\check{\mathfrak{W}}_{-\frac{3}{2}}^0, \check{\mathfrak{W}}_{-\frac{3}{2}}^1, \check{\mathfrak{W}}_{-\frac{1}{2}}^0$  and  $\check{\mathfrak{W}}_{-\frac{1}{2}}^1$ , respectively, described by

$$\begin{aligned} \mathfrak{W}_{-\frac{1}{2}}^1 &\cong \check{\mathfrak{W}}_{-\frac{1}{2}}^1 := \mathfrak{V}_{\frac{1}{2}} \otimes_{\circ} \left( \wedge^2 \mathfrak{V}_{-\frac{1}{2}} \right), & \mathfrak{W}_{-\frac{3}{2}}^1 &\cong \check{\mathfrak{W}}_{-\frac{3}{2}}^1 := \mathfrak{V}_{-\frac{1}{2}} \odot \left( \wedge^2 \mathfrak{V}_{-\frac{1}{2}} \right), \\ \mathfrak{W}_{-\frac{1}{2}}^0 &\cong \check{\mathfrak{W}}_{-\frac{1}{2}}^0 := \mathfrak{V}_{-\frac{1}{2}}, & \mathfrak{W}_{-\frac{3}{2}}^0 &\cong \check{\mathfrak{W}}_{-\frac{3}{2}}^0 := \wedge^3 \mathfrak{V}_{-\frac{1}{2}}. \end{aligned} \quad (4.2)$$

Here,  $\odot$  denotes the tracefree part of the tensor product. To obtain the Penrose diagram encoding the full  $\mathfrak{p}$ -invariance, we must also examine the action of the nilpotent part  $\mathfrak{g}_1$  of  $\mathfrak{p}$  on each of these irreducible  $\mathfrak{g}_0$ -modules. This is straightforward, and the result is given in Proposition 4.2.

**Remark 4.1** Extra care must be taken when  $m = 3$  where each of  $\wedge^3 \mathfrak{V}_{\frac{1}{2}}$  and  $\wedge^3 \mathfrak{V}_{-\frac{1}{2}}$  is one-dimensional. We can realise  $\mathfrak{g}_1$  as the pairing of  $\mathfrak{V}^{-\frac{1}{2}} / \mathfrak{V}^{\frac{1}{2}}$  and  $\wedge^3 \mathfrak{V}_{\frac{1}{2}}$ : any element of  $\mathfrak{g}_1$  can be written in the form  $\phi_{ab} = \frac{1}{2} \varepsilon_{abc} \phi^c$  for some vector  $\phi^c \in \mathfrak{V}_{-\frac{1}{2}}$ , where  $\varepsilon_{abc} \in \wedge^3 \mathfrak{V}_{\frac{1}{2}}$ . It then follows that  $\mathfrak{g}_1 \cdot \wedge^3 \mathfrak{V}_{-\frac{1}{2}} = \mathfrak{V}_{-\frac{1}{2}}$ .

Alternatively, and along the lines followed in the main text of this article, we define maps

$$\begin{aligned} \mathfrak{W} \Pi_{-\frac{3}{2}}^0(\Gamma) &:= \Gamma_{abc} \xi^{a(A} \xi^{bB)} \xi^{cC}, \\ \mathfrak{W} \Pi_{-\frac{3}{2}}^1(\Gamma) &:= \Gamma_{abc} \xi^{a[A} \xi^{bB} \xi^{cC]}, \\ \mathfrak{W} \Pi_{-\frac{1}{2}}^0(\Gamma) &:= \begin{cases} \Gamma_{abc} \xi^{bB} \xi^{cC} + \frac{2}{m-1} \left( \xi_a^{[B} \Gamma_{bcd} \xi^{cdD'} \gamma_{D'}^{b[C} \right] + \xi^{b[B} \Gamma_{bcd} \xi^{cdD'} \gamma_{aD'}^{C]} \right), & m > 3, \\ \Gamma_{abc} \xi^{bB} \xi^{cC} + \xi_a^{[B} \Gamma_{bcd} \xi^{cd} \gamma^{bD]C]} + \xi^{b[B} \Gamma_{bcd} \xi^{cd} \gamma_a^{D]C]} - \frac{8}{3} \xi^{bA} \Gamma_{bcd} \xi^{cd} \gamma_a^{BC}, & m = 3, \end{cases} \\ \mathfrak{W} \Pi_{-\frac{1}{2}}^1(\Gamma) &:= \xi^{A'} \Gamma_{bcd} \xi^{cdD'} \gamma_{D'}^{bB} + \xi^{bB} \Gamma_{bcd} \xi^{cdA'}, \end{aligned}$$

where  $\Gamma_{abc} \in \mathfrak{V} \otimes \mathfrak{g}$ . We have however abused notation for conciseness in the sense that the ‘ $\Gamma$ ’ in  $\mathfrak{W} \Pi_i^j(\Gamma)$  really denotes an element of  $\mathfrak{W}$ .

Note that in the light of Remark 4.1, one needs to distinguish the cases  $m = 3$  and  $m > 3$  in the definition of the map  $\mathfrak{W} \Pi_{-\frac{1}{2}}^0$ . For  $m = 3$ , we have made use of the isomorphism  $\mathfrak{S}^+ \cong (\mathfrak{S}^-)^*$ . Notationally, the primed indices are eliminated, and the  $\gamma$ -matrices take the form  $\gamma^{aAB}$  and  $\gamma^a_{AB}$  and are skew-symmetric in their spinor indices.<sup>7</sup>

<sup>7</sup>One may also use the identity

$$\xi^{a[A} \Gamma_{abc} \xi^{bB} \xi^{cC]} = -\frac{2}{3} \xi^{aE} \Gamma_{abc} \xi^{bc} \varepsilon^E{}^{ABC} = -\frac{1}{6} \gamma^a_{EF} \Gamma_{abc} \xi^{bcE} \xi_b^F \varepsilon^{ABC},$$

where  $\varepsilon^{ABC} := \frac{1}{2} \gamma^{aAB} \gamma_a^{CD} \xi_D$  is completely skew-symmetric.

**Proposition 4.2** *Each summand  $\mathrm{gr}_i(\mathfrak{W})$  of  $\mathrm{gr}(\mathfrak{W})$  of  $\mathfrak{W}$  decomposes into a direct sum*

$$\mathrm{gr}_{-\frac{1}{2}}(\mathfrak{W}) = \mathfrak{W}_{-\frac{1}{2}}^0 \oplus \mathfrak{W}_{-\frac{1}{2}}^1, \quad \mathrm{gr}_{-\frac{3}{2}}(\mathfrak{W}) = \mathfrak{W}_{-\frac{3}{2}}^0 \oplus \mathfrak{W}_{-\frac{3}{2}}^1,$$

*of irreducible  $\mathfrak{p}$ -modules, where*

$$\begin{aligned} \mathfrak{W}_{-\frac{3}{2}}^0 &= \{\Gamma \in \mathfrak{W}^{-\frac{3}{2}} : \mathfrak{W}\Pi_{-\frac{3}{2}}^1(\Gamma) = 0\} / \mathfrak{W}^{-\frac{1}{2}}, & \mathfrak{W}_{-\frac{3}{2}}^1 &= \{\Gamma \in \mathfrak{W}^{-\frac{3}{2}} : \mathfrak{W}\Pi_{-\frac{3}{2}}^0(\Gamma) = 0\} / \mathfrak{W}^{-\frac{1}{2}}, \\ \mathfrak{W}_{-\frac{1}{2}}^0 &= \{\Gamma \in \mathfrak{W}^{-\frac{1}{2}} : \mathfrak{W}\Pi_{-\frac{1}{2}}^1(\Gamma) = 0\}, & \mathfrak{W}_{-\frac{1}{2}}^1 &= \{\Gamma \in \mathfrak{W}^{-\frac{1}{2}} : \mathfrak{W}\Pi_{-\frac{1}{2}}^0(\Gamma) = 0\}. \end{aligned}$$

*Further the structure of  $\mathrm{gr}(\mathfrak{W})$  can be expressed by means of the graph*

$$\begin{array}{ccc} \mathfrak{W}_{-\frac{1}{2}}^1 & \longrightarrow & \mathfrak{W}_{-\frac{3}{2}}^1 \\ & \searrow \text{dotted} & \nearrow \\ \mathfrak{W}_{-\frac{1}{2}}^0 & \longrightarrow & \mathfrak{W}_{-\frac{3}{2}}^0 \end{array}$$

*where the dotted arrow occurs only when  $m > 3$ . Here, arrows are defined according to the relation*

$$\mathfrak{W}_i^j \longrightarrow \mathfrak{W}_{i-1}^k \iff \check{\mathfrak{W}}_i^j \subset \mathfrak{g}_1 \cdot \check{\mathfrak{W}}_{i-1}^k.$$

As before the arrows can be obtained by verifying the veracity of the statment

$$\text{if } \Gamma \in \mathfrak{W} \text{ satisfies } \mathfrak{W}\Pi_i^j(\Gamma) = 0 \text{ then } \mathfrak{W}\Pi_{i-1}^k(\Gamma) = 0.$$

**Remark 4.3** When  $m = 2$ , the story is similar if one defines  $\mathfrak{W} := \mathfrak{V} \otimes \mathfrak{g}^+$  where  $\mathfrak{g}^+ := \mathfrak{sl}(2, \mathbb{C})$  acts on  $\mathfrak{S}^+$ . In this case, one also obtains a filtration (4.1), but this time, each summand of the associated grading vector space is irreducible.

## 5 Differential geometry of pure spinor fields

Throughout this section,  $(\mathcal{M}, g_{ab})$  will denote an  $n$ -dimensional oriented complex Riemannian manifold,<sup>8</sup> where  $n = 2m$ , i.e. a complex manifold  $\mathcal{M}$  equipped with a global non-degenerate holomorphic section  $g_{ab}$  of  $\odot^2 T^* \mathcal{M}$ , where  $T^* \mathcal{M}$  is the holomorphic cotangent bundle of  $\mathcal{M}$ . The existence of a holomorphic Riemannian metric on  $\mathcal{M}$  is equivalent to a reduction of the structure of the frame bundle  $\mathcal{FM}$  of  $\mathcal{M}$  to  $G = \mathrm{SO}(2m, \mathbb{C})$ . Holomorphic vector bundles over  $\mathcal{M}$  can be constructed in terms of finite representations of  $G$  in the standard way [Sal89, ČS09]. For instance, if  $\mathfrak{V}$  is the standard representation of  $G$ , then the holomorphic tangent bundle is simply  $T\mathcal{M} := \mathcal{FM} \times_G \mathfrak{V}$ , and holomorphic sections of  $T\mathcal{M}$  can be viewed as equivariant holomorphic functions on  $\mathcal{FM}$  taking values in  $\mathfrak{V}$ .

The unique torsion-free metric-compatible holomorphic Levi-Civita connection and its associated covariant derivative on  $\mathcal{M}$  will both be denoted by  $\nabla_a$ . Adopting the notation of [PR84], we can make a choice of trivialisation by picking an orthonormal frame  $\{\delta_a^{\mathbf{a}}\}$  with dual  $\{\delta_a^{\mathbf{a}}\}$  where bold lower case Roman indices run from 1 to  $n$ , in which case the connection  $\nabla_a$  is represented by the  $\mathfrak{g}$ -valued 1-form  $\Gamma_{\mathbf{ab}}^{\mathbf{c}} := (\delta_a^{\mathbf{a}} \nabla_a \delta_b^{\mathbf{b}}) \delta_c^{\mathbf{c}}$ . From the viewpoint of [PR84], we can choose a connection  $\partial_a$  compatible with  $g_{ab}$ ,  $\{\delta_a^{\mathbf{a}}\}$  and  $\{\delta_a^{\mathbf{a}}\}$  so that

$$\nabla_a V^b = \partial_a V^b + \Gamma_{ac}^b V^c, \tag{5.1}$$

for any vector field  $V^a$ , where we have defined the tensor  $\Gamma_{ab}^c := \delta_a^{\mathbf{a}} \delta_b^{\mathbf{b}} \Gamma_{\mathbf{ab}}^{\mathbf{c}} \delta_c^{\mathbf{c}}$ . The torsion of  $\partial_a$  is in general non-vanishing and equals  $2\Gamma_{[ab]}^c$ . The Riemann tensor of  $\nabla_a$  is given by the identity

$$2\nabla_{[a} \nabla_{b]} V^d = R_{abc}^d V^c,$$

<sup>8</sup>There are topological obstructions for the existence of a holomorphic Riemannian metric [LeB83].

for any vector field  $V^a$ , and satisfies the Bianchi identity

$$\nabla_{[a} R_{b]c]de} = 0. \quad (5.2)$$

The Riemann tensor splits into  $O(2m, \mathbb{C})$ -irreducible components as

$$R_{abcd} = C_{abcd} + \frac{4}{n-2} \Phi_{[c|[a} g_{b]|d]} + \frac{2}{n(n-1)} R g_{c[a} g_{b]d}. \quad (5.3)$$

where  $C_{abcd}$  is the Weyl tensor,  $\Phi_{ab}$  the tracefree part of the Ricci tensor  $R_{ab} := R_{acb}{}^c$ , and  $R := R_a{}^a$  the Ricci scalar. For  $m > 2$ , this decomposition is also  $SO(2m, \mathbb{C})$ -irreducible, but when  $m = 2$ , the Weyl tensor splits into a self-dual part and an anti-self-dual part, each  $SL(2, \mathbb{C})$ -irreducible.

We now briefly recall some background on spinor geometry, which can be found in more details in [LM89]. We shall assume that  $(\mathcal{M}, g_{ab})$  is endowed with a spin structure, and denote the spinor bundle, the chiral positive and negative spinor bundle,  $\mathcal{S}$ ,  $\mathcal{S}^+$  and  $\mathcal{S}^-$  respectively. Sections of  $\mathcal{S}^+$  and  $\mathcal{S}^-$  will be denoted in the obvious way by means of the abstract index notation of section 2, e.g. by  $\xi^{A'}$  and  $\zeta^A$  and similarly for their dual.

The spin connection on  $\mathcal{S}$ ,  $\mathcal{S}^+$  and  $\mathcal{S}^-$  can be constructed canonically as a lift of the Levi-Civita connection, and will also be denoted  $\nabla_a$ . It has the property of preserving the Clifford module structure of  $\mathcal{S}$  in the sense that

$$\nabla_a (V^b \gamma_{bA'}{}^B \xi^{A'}) = (\nabla_a V^b) \gamma_{bA'}{}^B + V^b \gamma_{bA'}{}^B \nabla_a \xi^{A'},$$

for any vector field  $V^a$  and positive spinor field  $\xi^{A'}$ . Similar expressions apply to the other spinor bundles. The curvature of the spin connection is given by

$$2\nabla_{[a} \nabla_{b]} \xi^{A'} = -\frac{1}{4} R_{abcd} \gamma^{cd}{}_{B'}{}^{A'} \xi^{B'},$$

for any spinor fields  $\xi^{A'}$ , and similarly for spinors of other types. With a choice of trivialisation, the covariant derivative of a spinor  $\xi^{A'}$  is given by

$$\nabla_a \xi^{A'} = \partial_a \xi^{A'} - \frac{1}{4} \Gamma_{abc} \gamma^{bc}{}_{B'}{}^{A'} \xi^{B'}, \quad (5.4)$$

where  $\partial_a$  is the derivative operator used in (5.1).

## 5.1 Projective pure spinor bundles

We shall refer to a (positive) spinor field  $\xi^{A'}$  as *pure*, if at every point  $p \in \mathcal{M}$ , the spinor  $\xi^{A'}$  determines a totally null (self-dual)  $m$ -dimensional vector subspace

$$\mathcal{N}_p := \{X^a \in T_p \mathcal{M} : X^a \xi_a{}^{A'} = 0\}$$

of  $T_p \mathcal{M}$ , where we have written  $\xi_a{}^A := \xi^{B'} \gamma_{aB'}{}^A$ . In particular,  $\xi^{A'}$  satisfies the purity condition (2.7) or (2.8) at every point. We shall refer to  $\mathcal{N}$  as the  $\alpha$ -plane distribution associated to  $\xi^{A'}$ . Similar definitions apply to negative pure spinor fields and their associated  $\beta$ -plane distributions. We shall also refer to  $\alpha$ -plane and  $\beta$ -plane distributions as *almost null structures* on  $\mathcal{M}$  in line with the terminology introduced in [TC12]. Since we may not be interested in the scale of a pure spinor field, it is more convenient to consider the following bundle.

**Definition 5.1** The *projective positive pure spinor bundle*  $\text{Gr}_m^+(\mathcal{T}\mathcal{M}, g)$  over  $\mathcal{M}$  is the bundle with fiber over a point  $p$  of  $\mathcal{M}$  isomorphic to the  $\frac{1}{2}m(m-1)$ -dimensional family  $\text{Gr}_m^+(T_p \mathcal{M}, g)$  of  $\alpha$ -planes in  $T_p \mathcal{M}$ . The *projective negative pure spinor bundle*  $\text{Gr}_m^-(\mathcal{T}\mathcal{M}, g)$  is defined similarly with respect to  $\beta$ -planes.

Clearly, a positive pure spinor field  $\xi^{A'}$  determines a section of  $\text{Gr}_m^+(\text{TM}, g)$ . Note however that the bundles  $\text{Gr}_m^\pm(\text{TM}, g)$  do *not* require the existence of a spin structure on  $\mathcal{M}$ .

Now, let  $\xi^{A'}$  be a projective pure spinor field, i.e. a (global) section of  $\text{Gr}_m^+(\text{TM}, g)$ . This is equivalent to a reduction of the structure group of the frame bundle to  $P$ , the parabolic subgroup stabilising  $\xi^{A'}$ . Given such a reduction, it then makes sense to consider filtrations of vector bundles, together with their associated graded vector bundles, constructed from finite representations of  $P$  or of its Lie algebra  $\mathfrak{p}$ . For instance, the  $\mathfrak{p}$ -invariant filtration  $\{\mathfrak{C}^i\}$  on the space  $\mathfrak{C}$  of tensors with Weyl symmetries gives rise to a filtration of vector subbundles  $\mathcal{C}^i$  over  $\mathcal{M}$ , where  $\mathcal{C}^i := \mathcal{FM} \times_P \mathfrak{C}^i$ , and so does the story go for the associated graded  $\mathfrak{p}$ -modules  $\text{gr}_i(\mathfrak{C})$ , its irreducible modules  $\mathfrak{C}_i^j$  and the graded  $\mathfrak{g}_0$ -modules  $\mathfrak{C}_i$  in the obvious way and notation [ČS09]. We shall then recycle the notation of the previous sections of this article in this curved setting as the need arises.

**Remark 5.2** For explicit computation, it is convenient to introduce a spinor field  $\eta_{A'}$  dual to  $\xi^{A'}$ , and choose a (local) basis  $\{\delta^{a\mathbf{A}}\}$  and its dual  $\{\delta_{\mathbf{A}}^a\}$  annihilated by  $\xi^{A'}$  and  $\eta_{A'}$  respectively. Here bold upper case Roman indices run from 1 to  $m$ . We can then require that the connection  $\partial_a$  used in (5.1) and (5.4) be compatible with the null frame  $\{\delta^{a\mathbf{A}}, \delta_{\mathbf{A}}^a\}$ ,  $\xi^{A'}$  and  $\eta_{A'}$  – this requires a choice of scale for these spinors. As before, this can be expressed in abstract index notation, so that the connection 1-form  $\Gamma_{ab}{}^c$  of  $\nabla_a$  can be expressed as

$$\begin{aligned} \Gamma_{abc} = & \xi_a{}^A \xi_b{}^B \xi_c{}^C \Gamma_{ABC} + 2 \xi_a{}^A \xi_{[b}{}^B \eta_{c]C} \Gamma_{AB}{}^C + \xi_a{}^A \eta_{bB} \eta_{cC} \Gamma_A{}^{BC} \\ & + \eta_{aA} \xi_b{}^B \xi_c{}^C \Gamma_{BC}{}^A + 2 \eta_{aA} \xi_{[b}{}^B \eta_{c]C} \Gamma_{B}{}^A{}^C + \eta_{aA} \eta_{bB} \eta_{cC} \Gamma^{ABC}, \end{aligned} \quad (5.5)$$

where  $\Gamma_{ABC} = \Gamma_{A[BC]}$ ,  $\Gamma_A{}^{BC} = \Gamma_A{}^{[BC]}$ ,  $\Gamma^{ABC} = \Gamma^{A[BC]}$ ,  $\Gamma_{BC}{}^A = \Gamma_{[BC]}^A$ ,  $\Gamma_{AB}{}^C$  and  $\Gamma_B{}^A{}^C$  all carry  $\mathfrak{sl}(m, \mathbb{C})$ -indices. These are the abstract versions of the connection components  $\Gamma_{\mathbf{ABC}} := (\delta_{\mathbf{A}}^a \nabla_a \delta_{\mathbf{B}}^b) \delta_{\mathbf{C}}^b$ ,  $\Gamma_{\mathbf{AB}}{}^C := (\delta_{\mathbf{A}}^a \nabla_a \delta_{\mathbf{B}}^b) \delta_b^C$ , and so on in the obvious way.

### 5.1.1 The intrinsic torsion associated to a projective pure spinor field

Having singled out a projective pure spinor field  $\xi^{A'}$  on  $\mathcal{M}$ , in other words, a distribution  $\mathcal{N}$  of  $\alpha$ -planes on  $\mathcal{M}$ , or a  $P$ -structure on the frame bundle  $\mathcal{FM}$ , it remains to classify the various degrees of ‘integrability’ of the  $P$ -structure. The general theory expounded in [Sal89] tells us that the  $P$ -structure being integrable to first order, i.e. there exists a torsion-free connection compatible with the  $P$ -structure, is essentially equivalent to the pure spinor field  $\xi^{A'}$  being recurrent with respect to the Levi-Civita connection  $\nabla_a$ , i.e.

$$\nabla_a \xi^{B'} = \alpha_a \xi^{B'}, \quad (5.6)$$

for some 1-form  $\alpha_a$ . The recurrent spinor equation (5.6) can be more conveniently expressed as

$$(\nabla_a \xi^{bB}) \xi_b{}^C = 0. \quad (5.7)$$

which is also equivalent to the Levi-Civita connection being  $\mathfrak{p}$ -valued. Since the connection 1-form is in general a  $\mathfrak{g}$ -valued 1-form, measuring the extent to which the  $P$ -structure is integrable to first order is tantamount to decomposing the  $\mathfrak{p}$ -module  $\mathfrak{W} := \mathfrak{V} \otimes (\mathfrak{g}/\mathfrak{p})$  into irreducible  $\mathfrak{p}$ -modules as carried out in section 4. The obstruction to the existence of a torsion-free connection with respect to which  $\xi^{A'}$  is recurrent is known as the *intrinsic torsion* of the  $P$ -structure defined by  $\xi^{A'}$ , and can be identified with the tensorial expression

$$(\nabla_a \xi^{bB}) \xi_b{}^C \in \mathfrak{W}^{-\frac{1}{2}} \otimes \wedge^2 \mathfrak{G}^{\frac{m-2}{4}}. \quad (5.8)$$

Note that the tensor (5.8) clearly does not depend on the scaling of  $\xi^{A'}$ , and it is skew-symmetric in its spinor indices – this follows from taking the covariant derivative of the purity condition (2.7) on  $\xi^{A'}$ .



**Remark 5.3** Continuing on from Remark 5.2, the covariant derivative of  $\xi^{A'}$  is given by

$$\nabla_a \xi^{D'} = -\frac{1}{4} \eta_{aA} \Gamma^{ABC} \eta_{bB} \eta_{cC} \xi^{bcD'} - \frac{1}{4} \xi_a^A \Gamma_A^{BC} \eta_{bB} \eta_{cC} \xi^{bcD'} - \frac{1}{2} (\xi_a^A \Gamma_{AC}^C + \eta_{aA} \Gamma_A^A{}^C) \xi^{D'}. \quad (5.9)$$

so that

$$(\nabla_a \xi^{bB}) \xi_b^C = \eta_{aA} \Gamma^{ABC} + \xi_a^A \Gamma_A^{BC}. \quad (5.10)$$

We can then think of the skew-symmetric spinor indices of (5.8) as projecting out the  $\mathfrak{g}_{-1}$ -part of the connection 1-form  $\Gamma_{ab}^c$ . This makes contact with the argument and notation introduced in section 4.

Thus, the classification of the intrinsic torsion of the  $P$ -structure associated to a pure field  $\xi^{A'}$  boils down to an application of Proposition 4.2 to the covariant derivative of  $\xi^{A'}$ .

**Proposition 5.4** *Assume  $m > 2$ . Pointwise, the intrinsic torsion of the  $P$ -structure projects trivially into*

- $\mathfrak{W}_{-\frac{3}{2}}^1$  if and only if

$$(\xi^{a(A} \nabla_a \xi^{bB)}) \xi_b^C = 0, \quad (5.11)$$

- $\mathfrak{W}_{-\frac{3}{2}}^0$  if and only if

$$(\xi^{a[A} \nabla_a \xi^{bB)}) \xi_b^{C]} = 0, \quad (5.12)$$

- $\mathfrak{W}_{-\frac{1}{2}}^1$ ,  $\mathfrak{W}_{-\frac{3}{2}}^0$  and  $\mathfrak{W}_{-\frac{3}{2}}^1$  if and only if

$$(\nabla_a \xi^{bB}) \xi_b^C + \frac{2}{m-1} (\xi_a^{[B} \nabla_b \xi^{cC]} + \xi^{b[B} \nabla_b \xi_a^{C]}) = 0, \quad (5.13)$$

- $\mathfrak{W}_{-\frac{1}{2}}^0$ ,  $\mathfrak{W}_{-\frac{3}{2}}^0$  and  $\mathfrak{W}_{-\frac{3}{2}}^1$  if and only if

$$\xi^{A'} \nabla_b \xi^{bB} - \xi^{bB} \nabla_b \xi^{A'} = 0. \quad (5.14)$$

In addition, when  $m = 3$ , the intrinsic torsion of the  $P$ -structure projects trivially into  $\mathfrak{W}_{-\frac{1}{2}}^1$  and  $\mathfrak{W}_{-\frac{3}{2}}^1$  if and only if

$$(\nabla_a \xi^{bB}) \xi_b^C + (\xi_a^{[B} \nabla_b \xi^{cC]} + \xi^{b[B} \nabla_b \xi_a^{C]}) - \frac{8}{3} \xi^{bA} (\nabla_b \xi_A) \gamma_a^{BC} = 0, \quad (5.15)$$

where we have made use of the isomorphism  $\mathfrak{S}^+ \cong (\mathfrak{S}^-)^*$ .

This distinguishes eight, respectively seven, distinct  $\mathfrak{p}$ -invariant classes of the intrinsic torsion of the  $P$ -structure associated to  $\xi^{A'}$  when  $m = 3$ , respectively  $m > 3$ .

**Remark 5.5** When  $m = 2$ , by Remark 4.3, conditions (5.12) and (5.13) are vacuous, and one distinguishes only three classes of intrinsic torsion. The more familiar spinorial expressions in this case can be found in appendix A.1.

**Remark 5.6** For the case  $m = 3$ , we refer to appendix A.2 where conditions (5.11), (5.12), (5.15) and (5.14) are given as conditions (A.6), (A.7), (A.8) and (A.9).

**Remark 5.7** In terms of the Levi-Civita connection 1-form (5.5), conditions (5.12), (5.11), (5.13) and (5.14) are equivalent to

$$\begin{aligned}\Gamma^{[ABC]} &= 0, \\ \Gamma^{(AB)C} &= 0, \\ \Gamma_A^{BC} - \frac{2}{m-1} I_A^{[B} \Gamma_D^{D]C]} &= 0, & \Gamma^{ABC} &= 0, \\ \Gamma_A^{AC} &= 0, & \Gamma^{ABC} &= 0,\end{aligned}$$

respectively. When  $m = 3$ , condition (5.15) is equivalent to

$$\Gamma_A^{BC} - \frac{2}{m-1} I_A^{[B} \Gamma_D^{D]C]} = 0, \quad \Gamma^{(AB)C} = 0.$$

### 5.1.2 Geometric properties

The following lemma is an immediate consequent of the above discussion.

**Lemma 5.8** *The following three (pointwise) statements are equivalent:*

- the intrinsic torsion of the  $P$ -structure lies in the  $\mathfrak{p}$ -module  $\mathfrak{W}^{-\frac{1}{2}}$ ;
- the spinor field satisfies

$$(\xi^{aA} \nabla_a \xi^{bB}) \xi_b^C = 0; \quad (5.16)$$

- the connection 1-form takes values in  $\mathfrak{p}$  along  $\mathcal{N}$ .

We can now give a geometrical interpretation of the differential condition (5.16).

**Proposition 5.9** *A projective pure spinor field  $\xi^{A'}$  (locally) satisfies equation (5.16) if and only if its associated almost null structure on  $\mathcal{M}$  is (locally) integrable, i.e.  $[\Gamma(\mathcal{N}), \Gamma(\mathcal{N})] \subset \Gamma(\mathcal{N})$ .*

*Proof.* The integrability of the almost null structure  $\mathcal{N}$  is equivalent to  $g_{ab} X^a Y^c \nabla_c Z^b = 0$ , for all  $X^a, Y^a, Z^a$  in  $\Gamma(\mathcal{N})$  as shown in [TC12]. In particular, we have that the connection 1-form takes values in  $\mathfrak{p}$  along  $\mathcal{N}$ . The result follows immediately by Lemma 5.8.  $\square$

**Definition 5.10** We say that a pure (projective) spinor field  $\xi^{A'}$  is *foliating* if it satisfies equation (5.16).

**Conformal invariance** Adopting the conventions given in Appendix B and setting  $\hat{\xi}_a^A := \xi^{B'} \hat{\gamma}_{aB'}^A$ , we compute

$$\begin{aligned}(\hat{\nabla}_a \hat{\xi}^{bB}) \hat{\xi}_b^C &= (\nabla_a \xi^{bB}) \xi_b^C - 2\Upsilon_b \xi^{b[B} \xi_a^{C]}, \\ \xi^{A'} \hat{\nabla}_b \hat{\xi}^{bB} - \hat{\xi}^{bB} \hat{\nabla}_b \xi^{A'} &= \Omega^{-1} \left( \xi^{A'} \nabla_b \xi^{bB} - \xi^{bB} \nabla_b \xi^{A'} + (m-1) \Upsilon_a \xi^{aB} \xi^{A'} \right), \\ (\hat{\xi}^{bA} \hat{\nabla}_a \hat{\xi}^{bB}) \hat{\xi}_b^C &= \Omega^{-1} (\xi^{aA} \nabla_a \xi^{bB}) \xi_b^C.\end{aligned}$$

The first two equations can be combined to yield

$$(\hat{\nabla}_a \hat{\xi}^{bB}) \hat{\xi}_b^C + \frac{2}{m-1} \left( \hat{\xi}_a^{[B} \hat{\nabla}_b \hat{\xi}^{bC]} + \hat{\xi}^{b[B} \hat{\nabla}_b \hat{\xi}_a^{C]} \right) = (\nabla_a \xi^{bB}) \xi_b^C + \frac{2}{m-1} \left( \xi_a^{[B} \nabla_b \xi^{bC]} + \xi^{b[B} \nabla_b \xi_a^{C]} \right).$$

The following proposition is now immediate.

**Proposition 5.11** Assume  $m > 2$ . The conditions (5.11), (5.12) and (5.13) are conformally invariant. Suppose further that  $\xi^{A'}$  is a pure spinor field satisfying (5.13) and

$$\xi^{A'} \nabla_b \xi^{bB} - \xi^{bB} \nabla_b \xi^{A'} = -(m-1) \xi^{A'} \xi^{bB} \nabla_b f, \quad (5.17)$$

for some complex analytic function  $f$ . Then  $(\mathcal{M}, g_{ab})$  is (locally) conformal to a complex Riemannian manifold admitting a recurrent pure spinor field.

**Remark 5.12** Proposition 5.11 can be thought of as a complex Riemannian analogue of Hermitian manifolds locally conformal to Kähler [FFS94].

### Curvature conditions

**Lemma 5.13** Let  $\xi^{A'}$  be a foliating pure spinor field. Then

$$\xi^{aA} \xi^{bB} \xi^{cC} \xi^{dD} C_{abcd} = 0. \quad (5.18)$$

*Proof.* We first note that the foliating spinor equation (5.16) can be rewritten as

$$\xi^{aA} \nabla_a \xi^{B'} = \alpha^A \xi^{B'},$$

for some  $\alpha^A$ . Differentiating it along  $\mathcal{N}$  yields

$$\alpha^A \alpha^B \xi^{C'} + \xi^{aA} \xi^{bB} \nabla_a \nabla_b \xi^{C'} = (\xi^{aA} \nabla_a \alpha^B) \xi^{C'} + \alpha^A \alpha^B \xi^{C'}.$$

Commuting the derivatives leads to

$$-\frac{1}{4} R_{abcd} \xi^{aA} \xi^{bB} \gamma^{cd}{}_{D'}{}^{C'} \xi^{D'} = (\xi^{a[A} \nabla_a \alpha^{B]}) \xi^{A'}, \quad (5.19)$$

which is equivalent to  $\xi^{aA} \xi^{bB} \xi^{cC} \xi^{dD} R_{abcd} = 0$ . The decomposition of the Riemann tensor together with the purity condition concludes the proof.  $\square$

**Remark 5.14** When  $m = 2$ , most of the differential equations on a *positive* spinor field lead to integrability conditions, such as (5.18), which can be seen to restrict to the *self-dual* Weyl curvature only.

**Lemma 5.15** Let  $\xi^{A'}$  be a recurrent pure spinor field. Then

$$\xi^{aA} \xi^{bB} R_{abcd} = 0, \quad (5.20)$$

$$\xi^{aA} \xi^{bB} \Phi_{ab} = \xi^{aA} \xi^{bB} R_{ab} = 0, \quad (5.21)$$

$$\xi^{aA} \xi^{bB} \xi^{cC} C_{abcd} = 0, \quad (5.22)$$

and in addition, when  $m > 3$ ,

$${}^{\mathfrak{e}}\Pi_0^2(C) = 0. \quad (5.23)$$

Further,

$$R = 0 \quad \Longleftrightarrow \quad \xi^{abA'} C_{abcd} \xi^{cdD'} = 0.$$

and in addition, when  $m > 2$ ,

$$\xi^{aA} \Phi_{ab} = 0 \quad \Longleftrightarrow \quad {}^{\mathfrak{e}}\Pi_0^1(C) = 0.$$

Here,  ${}^{\mathfrak{e}}\Pi_0^1$  and  ${}^{\mathfrak{e}}\Pi_0^2$  are defined in (3.5).

*Proof.* Taking a covariant derivative of equation (5.6) and commuting the derivatives yield

$$-\frac{1}{4}R_{abcd}\gamma^{cd}{}_{B'}{}^{A'}\xi^{B'} = (\nabla_{[a}\alpha_{b]})\xi^{A'},$$

which is equivalent to equation (5.20). Contracting equation (5.20) with  $\gamma^{ab}{}_B{}^C$  yields the condition (5.21) on the Ricci tensor. Finally, conditions (5.22) and (5.23) on the Weyl tensor is obtained from the decomposition of the Riemann tensor (5.3) and (B.3). We find

$$\begin{aligned}\xi^{aA}C_{a[bc]d}\xi^{dD} &= \frac{2}{n-2}\xi_{[b}{}^{[A}\Phi_{c]d}\xi^{dD]} + \frac{1}{n(n-1)}R\xi_{[b}{}^A\xi_{c]}{}^D, \\ \xi^{aeC'}C_{aedb}\xi^{dD} &= 2\frac{n-4}{n-2}\xi^{C'}\Phi_{bd}\xi^{dD} + 2\frac{n-2}{n(n-1)}R\xi_b{}^D\xi^{C'}, \\ \xi^{aeC'}C_{aedf}\xi^{dfF'} &= -2\frac{n-2}{n-1}R\xi^{C'}\xi^{F'},\end{aligned}$$

and the remaining statements follow immediately from the formulae for  ${}^{\mathfrak{C}}\Pi_0^2$ ,  ${}^{\mathfrak{C}}\Pi_0^1$ ,  ${}^{\mathfrak{C}}\Pi_0^0$ .  $\square$

**Remark 5.16** The purity condition is crucial in deducing conditions (5.22) and (5.23) on the Weyl tensor.

## 5.2 Spinorial differential equations

So far we have only considered spinorial differential equations on *projective* pure spinor fields, i.e. differential equations that are invariant under rescalings of  $\xi^{A'}$ . In this section, we study spinorial differential equations on pure spinors of fixed scales emphasizing their relations to the intrinsic torsion of their associated  $P$ -structures.

### 5.2.1 Scaled foliating spinors

A simple variation of the foliating spinor equation (5.16) is given by the stronger condition

$$\xi^{aA}\nabla_a\xi^{B'} = 0. \quad (5.24)$$

Since  $\hat{\xi}^{aB}\hat{\nabla}_a(\Omega^{-1}\xi^{A'}) = \Omega^{-2}(\xi^{aB}\nabla_a\xi^{A'})$ , equation (5.24) is clearly conformally invariant if and only if the spinor field  $\xi^{A'}$  has conformal weight  $-1$ . Accordingly, the integrability condition for (5.24) is expected to be conformally invariant. Indeed, a variation of the proof of Lemma 5.13 with  $\alpha_a = 0$  leads to

**Lemma 5.17** *Let  $\xi^{A'}$  be a pure spinor field satisfying equation (5.24). Then  $C_{abcd}\xi^{aA}\xi^{bB}\xi^{cdC'} = 0$ .*

### 5.2.2 Parallel pure spinors

Arguably, the most restrictive differential condition on a spinor field  $\xi^{A'}$  is that it is parallel, i.e.

$$\nabla_a\xi^{B'} = 0. \quad (5.25)$$

Such a condition has been studied in the context of pseudo-Riemannian geometry, and is well-known to be related to holonomy reduction of the Levi-Civita connection [Wan89, Bry00]. The integrability condition for (5.25) can be obtained from the proof of Lemma 5.15 by setting  $\alpha_a = 0$ .

**Lemma 5.18** *Let  $\xi^{A'}$  be a parallel (not necessarily pure) spinor. Then*

$$\begin{aligned}R_{abcd}\xi^{cdD'} &= 0, \\ \Phi_{ab}\xi^{bB} &= R_{ab}\xi^{bB} = 0 = R, \\ C_{abcd}\xi^{cdD'} &= 0.\end{aligned}$$

The above result is standard. Normal forms of metrics admitting parallel (pure) spinors can be found in [Bry00, Dun02].

### 5.2.3 Simple zero-rest-mass fields

Conditions weaker than (5.25) can be obtained by decomposing the covariant derivative of a spinor field into two irreducible parts under  $\text{Spin}(2m, \mathbb{C})$ . The smaller of these is known as the *(Weyl-)Dirac equation*

$$\gamma^a_{A'}{}^B \nabla_a \xi^{A'} = 0, \quad (5.26)$$

and it admits a generalisation to irreducible symmetric spinor fields of higher valence, i.e. spinor fields<sup>9</sup>  $\phi^{A'_1 A'_2 \dots A'_k} = \phi^{(A'_1 A'_2 \dots A'_k)}$  satisfying  $\gamma^a_{A'_1}{}^B \gamma_{a A'_2}{}^C \phi^{A'_1 A'_2 \dots A'_k} = 0$ , which is known as the *zero-rest-mass (zrm) field equation*,

$$\gamma^a_{A'_1}{}^B \nabla_a \phi^{A'_1 A'_2 \dots A'_k} = 0. \quad (5.27)$$

The case  $k = 2$  corresponds to a closed, and thus coclosed, self-dual  $m$ -form. Equations (5.27) is conformally invariant provided that its solutions  $\phi^{A'_1 A'_2 \dots A'_k}$  are of conformal weight  $-m - k + 1$  respectively. For  $k > 2$ , there is a strong integrability condition on  $\phi^{A'_1 A'_2 \dots A'_k}$  given by the following lemma.

**Lemma 5.19** *For  $k > 2$ , let  $\phi^{A'_1 A'_2 \dots A'_k}$  be a solution of the zrm field equation (5.27). Then*

$$\gamma^a_{C'_1}{}^A \gamma^b_{C'_2}{}^B C_{abcd} \gamma^{cd}_{D'} (C'_3 \phi^{C'_4 \dots C'_k}) C'_1 C'_2 D' = 0.$$

*Proof.* We compute

$$\begin{aligned} 0 &= 2\gamma^a_{C'_1}{}^A \gamma^b_{C'_2}{}^B [\nabla_a \nabla_b \phi^{C'_1 C'_2 \dots C'_k} \\ &= -\frac{1}{2} \gamma^a_{C'_1}{}^A \gamma^b_{C'_2}{}^B R_{abcd} \gamma^{cd}_{D'} C'_1 \phi^{C'_2 \dots C'_k D'} - \frac{k-2}{4} \gamma^a_{C'_1}{}^A \gamma^b_{C'_2}{}^B R_{abcd} \gamma^{cd}_{D'} (C'_3 \phi^{C'_4 \dots C'_k}) C'_1 C'_2 D' \\ &= \Phi_{bc} \gamma^b_{C'_2}{}^A \gamma^c_{D'}{}^B \phi^{C'_2 \dots C'_k D'} - \frac{k-2}{4} \gamma^a_{C'_1}{}^A \gamma^b_{C'_2}{}^B C_{abcd} \gamma^{cd}_{D'} (C'_3 \phi^{C'_4 \dots C'_k}) C'_1 C'_2 D'. \end{aligned}$$

The first term must vanish by symmetry consideration, which concludes the proof.  $\square$

An irreducible symmetric spinor as above is said to be *simple* if it takes the form

$$\phi^{A'_1 A'_2 \dots A'_k} = e^\psi \xi^{A'_1} \xi^{A'_2} \dots \xi^{A'_k}, \quad (5.28)$$

for some (complex analytic) function  $\psi$  and pure spinor field  $\xi^{A'}$ . In this case, the integrability condition for the existence of a solution of equation (5.27) is given by the following

**Corollary 5.20** *For  $k > 2$ , let  $\phi^{A'_1 A'_2 \dots A'_k}$  be a simple solution (5.28) of the zrm field equation (5.27). Then*

$$\xi^{aA} \xi^{bB} C_{abcd} \xi^{cdD'} = 0.$$

The relation between pure solutions to the zrm field equation and the foliating condition was first established by Robinson [Rob61] in four dimensions in his study of electromagnetism. It was later generalised by Hughston and Mason [HM88] to even dimensions.

**Theorem 5.21 (Robinson (1961), Hughston & Mason (1988))** *Let  $\phi^{A'_1 \dots A'_k}$  be a simple spinor field of the form (5.28). Suppose that  $\phi^{A'_1 \dots A'_k}$  is a solution to the zrm field equation (5.27). Then  $\xi^{A'}$  is foliating.*

*Conversely, suppose that  $\xi^{A'}$  is a foliating pure spinor. When  $k > 2$ , suppose further that*

$$\xi^{aA} \xi^{bB} C_{abcd} \xi^{cdD'} = 0.$$

*Then there exists a complex analytic function  $\psi$  such that the spinor field  $\phi^{A'_1 A'_2 \dots A'_k}$  of the form (5.28) satisfies the zrm field equation (5.27).*

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<sup>9</sup>From a representational theoretic viewpoint,  $\phi^{A'_1 A'_2 \dots A'_k}$  lies in the  $k$ -fold Cartan product of the positive spinor representation.

### 5.2.4 Twistor-spinors

The larger irreducible part of the covariant derivative of a spinor field leads to the *twistor equation*

$$\nabla_a \xi^{A'} + \frac{1}{n} \gamma_{aB}^{A'} \zeta^B = 0, \quad (5.29)$$

which determines  $\zeta^B := \gamma_{A'}^B \nabla_a \xi^{A'}$ . We shall refer to its solution  $\xi^{A'}$  as a *twistor-spinor*. It is well-known that the twistor-spinor equation is overdetermined, and for this reason, it is often more convenient to consider its prolongation [PR86, BJ10]

$$\nabla_a \zeta^B + \frac{1}{2} P_{ab} \gamma_{A'}^B \zeta^{A'} = 0, \quad (5.30)$$

where  $P_{ab}$  is the *Rho* or *Schouten* tensor defined in appendix B. We immediately deduce

$$\nabla_b \zeta^{bA'} = -\frac{n}{4(n-1)} R \xi^{A'}. \quad (5.31)$$

Equations (5.29) and (5.30) are conformally invariant, and this property is best encoded by regarding the equivalence class of pair of spinors  $(\xi^{A'}, \zeta^A) \sim (\hat{\xi}^{A'}, \hat{\zeta}^A)$  given by

$$\xi^{A'} \mapsto \hat{\xi}^{A'} = \xi^{A'}, \quad \zeta^A \mapsto \hat{\zeta}^A = \Omega^{-1} \left( \zeta^A + \frac{n}{2} \Upsilon_a \xi^{aA'} \right). \quad (5.32)$$

as a section of the *local twistor bundle* [PR86] or *spin tractor bundle* [HS11]. This bundle arises from a chiral spinor representation for  $\text{Spin}(2m+2, \mathbb{C})$ .

For future reference, we record the integrability condition for a twistor-spinor [BJ10].

**Lemma 5.22** *Let  $\xi^{A'}$  be a twistor-spinor as above. Then*

$$C_{abcd} \xi^{cdB'} = 0, \quad C_{abcd} \zeta^{cdC} - 2n A_{cab} \xi^{cC} = 0, \quad A_{bcd} \xi^{cdA'} = 0.$$

where  $A_{abc}$  is the Cotton-York tensor defined in appendix B.

We shall mostly be concerned with the case where the twistor-spinor  $\xi^{A'}$  is pure. We note that the purity of  $\xi^{A'}$  does not in general entail the purity of  $\zeta^A$  in dimensions greater than six.

In four dimensions, a twistor-spinor is always foliating [PR86], but it is not so in general in higher dimensions. We can nevertheless give necessary and sufficient conditions for this to happen.

**Proposition 5.23** *Assume  $m > 2$ . Let  $\xi^{A'}$  be a pure twistor-spinor, and let  $\zeta^B := \gamma_{A'}^B \nabla_a \xi^{A'}$ . Then  $\xi^{A'}$  satisfies*

$$(\xi^{a(A} \nabla_a \xi^{bB)}) \zeta_b^{C'} = 0. \quad (5.11)$$

Further,  $\xi^{A'}$  is foliating if and only if  $\zeta^A$  is pure and the  $\beta$ -plane distribution associated to  $\zeta^A$  intersects the  $\alpha$ -plane distribution associated to  $\xi^{A'}$  in a totally null  $(m-1)$ -distribution, i.e.

$$\xi^{aB} \zeta_a^{A'} = -2 \zeta^B \xi^{A'}. \quad (5.33)$$

Suppose that  $\xi^{A'}$  is foliating so that  $\zeta^A$  is pure and satisfies conditions (5.33). Then

$$\left( \nabla_a \zeta^{bB'} \right) \zeta_b^{C'} = -2n P_{ab} \zeta^{b[B'} \xi^{C']}, \quad \left( \zeta^{aA'} \nabla_a \zeta^{bB'} \right) \zeta_b^{C'} = -2n P_{ab} \zeta^{aA'} \zeta^{b[B'} \xi^{C']}.$$

In particular,

$$\left( \zeta^{a[A'} \nabla_a \zeta^{bB'} \right) \zeta_b^{C'} = 0,$$

and  $\zeta^A$  is foliating, respectively recurrent, if and only if  $P_{ab} \zeta^{aA'} \zeta^{b[B'} \xi^{C']} = 0$ , respectively,  $P_{ab} \zeta^{b[B'} \xi^{C']} = 0$ .

*Proof.* From the twistor equation (5.29) and assuming  $\xi^{A'}$  to be pure, we compute

$$(\xi^{aA} \nabla_a \xi^{bB}) \xi_b{}^C = -\frac{1}{n} \xi^{aA} \zeta_a{}^{D'} \gamma^b{}_{D'}{}^B \xi_b{}^C = -\frac{1}{n} \xi^{aA} \zeta_{ab}{}^B \xi^{bC}.$$

This expression is skew in  $BC$  and  $AB$ , which proves the first claim.

To prove the second statement, it is easier to consider the contraction of equation (5.29) with  $\xi^{aA}$ , i.e.

$$\xi^{aB} \nabla_a \xi^{A'} = -\frac{1}{n} \xi^{aB} \zeta_a{}^{A'} =: O^{BA'}. \quad (5.34)$$

From an algebraic viewpoint,  $O^{BA'}$  lies in  $\mathfrak{S}_{\frac{m-2}{4}} \otimes (\mathfrak{S}_{\frac{m}{4}} \oplus \mathfrak{S}_{\frac{m-4}{4}})$ . But  $\xi^{A'}$  is foliating if and only if the Levi-Civita connection takes values in  $\mathfrak{p}$  along  $\mathcal{N}$ , i.e.  $O^{BA'}$  lies in  $\mathfrak{S}_{\frac{m-2}{4}} \otimes \mathfrak{S}_{\frac{m}{4}}$ . This means that the RHS of (5.34) factorises as in (2.11). Both the purity of  $\zeta^A$  and the geometric interpretation then follow from Proposition 2.9.

Finally, if  $\xi^{A'}$  is foliating, then  $\zeta^A$  is a  $\mathfrak{S}_{\frac{m-2}{4}}$ -valued function on  $\mathcal{M}$  by the above argument. A straightforward computation then leads to

$$\left( \nabla_a \zeta^{bB'} \right) \zeta_b{}^{C'} = -2n P_{ab} \zeta^{b[B'} \xi^{C']},$$

from which the remaining claims can be deduced.  $\square$

**Remark 5.24** An equivalent way of expressing the foliation condition of a twistor-spinor, i.e.  $\xi^{A'}$  and  $\zeta^A$  pure and satisfying (2.11), is that the *pair*  $(\xi^{A'}, \zeta^A)$  is a pure section of the local twistor bundle, i.e. it is a pure spinor for  $\text{Spin}(2m+2, \mathbb{C})$ .

**Remark 5.25** Clearly, if  $(\mathcal{M}, g_{ab})$  is Ricci-flat, then  $\zeta^A$  is parallel regardless of any purity condition on  $\xi^{A'}$  or  $\zeta^A$ .

When  $m = 3$ , Proposition 5.23 can be refined further.

**Proposition 5.26** *Let  $\xi_A$  be a positive twistor-spinor on a six-dimensional complex Riemannian manifold  $(\mathcal{M}, g_{ab})$ . Then  $\xi_A$  satisfies condition (5.15) (or (A.8)), and thus condition (5.11) (or (A.6)).*

This result can easily be proved using spinor calculus as is shown at the end of appendix A.2.

**Remark 5.27** It is interesting to point out that there are examples of non-foliating twistor-spinors on six-dimensional smooth pseudo-Riemannian manifolds of split signatures. It is shown in [Bry06] how one can associate to a generic 3-distribution  $\mathcal{N}$  on a six-dimensional manifold  $\mathcal{M}$  a conformal structure. By generic, we mean that  $\mathcal{N}$  is maximally non-integrable, i.e.  $\Gamma(\mathcal{N}) + [\Gamma(\mathcal{N}), \Gamma(\mathcal{N})] = \Gamma(T\mathcal{M})$ . The authors of [HS11] later characterised such conformal structures in terms of a twistor-spinor  $(\xi_A, \zeta^A)$  which is *generic* in the sense that  $(\xi_A, \zeta^A)$  satisfy the non-degeneracy condition  $\xi_A \zeta^A \neq 0$ . One views the pair  $(\xi_A, \zeta^A)$  as an ‘impure’ spinor for the group  $\text{Spin}(4, 4)$ , which is tantamount to the reduction of the conformal holonomy to (a subgroup of)  $\text{Spin}(3, 4)$ .

In the light of Proposition 5.26, there is a rather puzzling feature in this construction. From the perspective of Riemannian or conformal geometry, the intrinsic torsion of a *generic* non-integrable  $\alpha$ -plane distribution  $\mathcal{N}$  would lie in the whole of the  $\mathfrak{p}$ -module  $\mathfrak{W}^{-\frac{3}{2}}$ . In other words, if  $\xi^{A'}$  is the projective pure spinor field associated to  $\mathcal{N}$ , neither condition (5.11) nor (5.12) would hold. However, a generic 3-distribution  $\mathcal{N}$  on a six-dimensional smooth manifold  $\mathcal{M}$  becomes ‘less generic’ once it is identified as an  $\alpha$ -plane distribution with respect to the conformal structure of [Bry06, HS11] in the sense that its intrinsic torsion satisfies (5.11) (or (A.6)).

Next, we note that a parallel pure spinor is necessarily a foliating pure twistor-spinor. The following proposition in fact gives a partial local converse, thereby generalising a result in [Dun02] from four to higher even dimensions.

**Proposition 5.28** *Let  $\xi^{A'}$  be a foliating pure twistor-spinor on  $(\mathcal{M}, g_{ab})$  and suppose further that the metric can be rescaled so that the Schouten tensor satisfies  $P_{ab}\xi^{aA} = 0$ . In particular  $(\mathcal{M}, g_{ab})$  is conformally scalar-flat. Then there exists a conformal rescaling such that  $\xi^{A'}$  is locally parallel.*

*Proof.* Let  $\xi^{A'}$  be a twistor-spinor as in equation (5.29) with  $\zeta^A := \nabla_a \xi^{aA}$ . The integrability condition for the existence of a complex analytic function  $\phi$  such that

$$\zeta^A = \xi^{aA} \nabla_a \phi, \quad (5.35)$$

is that  $\xi^{A'}$  is foliating and [PR86, HM88]

$$\xi^{a[A} \nabla_a \zeta^{B]} = \alpha^{[A} \zeta^{B]}, \quad (5.36)$$

where  $\alpha^A$  is defined by  $\xi^{aA} \nabla_a \xi^{B'} = \alpha^A \xi^{B'}$ . By equation (5.30), the LHS of equation (5.36) vanishes, while equation (5.29) gives  $\alpha^A = \frac{2}{n} \zeta^A$ . Hence (5.36) holds. By virtue of (5.31),  $\phi$  will satisfy

$$\frac{n-2}{2} (\nabla^a \phi) (\nabla_a \phi) + \nabla^a \nabla_a \phi = 0.$$

Next, using the transformations (5.32), we show that one can choose a complex analytic conformal rescaling  $\Omega$  such that  $\hat{\zeta}^A = 0$  so that  $\hat{\xi}^{A'}$  becomes parallel in this scale. Using the potential form (5.35) for  $\zeta^A$ , this is equivalent to finding  $\Omega$  such that

$$\xi^{aA} \nabla_a \left( \phi + \frac{n}{2} \ln \Omega \right) = 0. \quad (5.37)$$

But (5.37) is satisfied by choosing  $\Omega = e^{-\frac{2}{n}\phi}$ . Such a choice is not unique since adding any complex analytic function annihilated by the vector fields tangent to the foliating defined by  $\xi^{A'}$  leaves (5.37) unchanged.

Finally, we can check that such a choice modulo any conformal change along  $\xi^{A'}$  preserves the condition  $P_{ab}\xi^{bA} = 0$  by considering the transformation rules for the Rho tensor (B.4).  $\square$

**Flat case** The prolongation of (5.29) gives an upper bound for the dimension of the space of its solutions of  $2^m$ , which is attained on conformally flat manifolds. For a  $2m$ -dimensional flat manifold  $(\mathbb{M}, g_{ab})$ , the twistor equation can be integrated explicitly

$$\xi^{A'} = \mathring{\xi}^{A'} + x^a \gamma_{aB}{}^{A'} \mathring{\zeta}^B, \quad \zeta^B = \mathring{\zeta}^B, \quad (5.38)$$

for some constant spinors  $\mathring{\xi}^{A'}$  and  $\mathring{\zeta}^B$ . The space of solutions  $(\mathring{\xi}^{A'}, \mathring{\zeta}^B)$  is thus a  $2^m$ -dimensional vector space, which can be identified as a chiral spinor representation for  $\text{Spin}(2m+2, \mathbb{C})$  covering the conformal group of transformations of the compactification of  $\mathbb{M}$ . In fact such a ‘conformal’ spinor is pure if and only if  $\mathring{\xi}^{A'}$  and  $\mathring{\zeta}^B$  are pure and satisfy the algebraic constraint  $\mathring{\xi}^{aA} \mathring{\zeta}_a{}^{B'} = -2\mathring{\xi}^{B'} \mathring{\zeta}^A$ . The space of such pure solutions to the twistor equations has then a nice geometrical interpretation. Its projectivation is the space of all  $\beta$ -planes in the compactification of  $\mathbb{M}$ , also known as the *primed twistor space* of  $\mathbb{M}$ .

Now assuming  $m \geq 4$ , one can show using (5.38) that  $\xi^{A'}$  is pure, and  $\xi^{aA} \mathring{\zeta}_a{}^{B'} = -2\xi^{B'} \mathring{\zeta}^A$  if and only if  $(\mathring{\xi}^{A'}, \mathring{\zeta}^A)$  is a pure conformal spinor. This means in particular any pure twistor-spinor field in flat space for  $m \geq 4$  is foliating.

However, in six dimensions, when  $m = 3$ , all spinors are pure, i.e.  $\xi^{A'}$ ,  $\mathring{\xi}^{A'}$  and  $\mathring{\zeta}^A$  are pure. The condition  $\mathring{\xi}_A \mathring{\zeta}^A = 0$  must therefore be an additional constraint on the twistor-spinor, which we know by Proposition 5.23 is a necessary and sufficient condition for the spinor  $\xi_A$  to be foliating. Thus, in six-dimensional flat space, not every (pure) twistor-spinor is foliating.



### 5.2.5 Relation to the Goldberg-Sachs theorem in higher dimensions

In [TC12], a higher-dimensional generalisation of the complex Goldberg-Sachs theorem is presented. Suppose that the Weyl tensor and the Cotton-York tensor satisfy the algebraic degeneracy conditions

$$C_{abcd}\xi^{aA}\xi^{bB}\xi^{cC} = 0, \quad A_{abc}\xi^{aA}\xi^{bB}\xi^{cC} = 0, \quad (5.39)$$

i.e. the Weyl tensor and the Cotton-York tensor have components in  $\mathfrak{C}^0$  and  $\mathfrak{A}^{\frac{1}{2}}$  respectively. Suppose further that the Weyl tensor is otherwise *generic*, then  $\xi^{A'}$  is foliating. A pure twistor-spinor will clearly satisfy the conditions (5.39) *except* for the genericity assumption, since as a consequence of Lemma 5.22,  ${}^{\mathfrak{C}}\Pi_1^0(C) = 0$ , and in particular,  ${}^{\mathfrak{C}}\Pi_0^0(C) = {}^{\mathfrak{C}}\Pi_0^1(C) = {}^{\mathfrak{C}}\Pi_0^2(C) = 0$ . However, by Proposition 5.23 a twistor-spinor is not foliating in general, but satisfies the weaker condition (5.11). This result thus highlights the importance of the genericity assumption in the formulation of the higher-dimensional version of the Goldberg-Sachs theorem. It would certainly be interesting to characterise the degeneracy of the Weyl tensor further to determine which complex Riemannian manifolds admit foliating twistor-spinors, and in fact this suggests the following conjecture generalising and improving the Goldberg-Sachs theorem of [TC12].

**Conjecture 5.29** *Suppose that  $\xi^{A'}$  is a pure spinor field on a  $2m$ -dimensional Einstein complex Riemannian manifold  $(\mathcal{M}, g_{ab})$  such that the Weyl tensor satisfies  $C_{abcd}\xi^{aA}\xi^{bB}\xi^{cC} = 0$ . Then  $\xi^{A'}$  satisfies*

$$\left(\xi^{a(A}\nabla_a\xi^{bB)}\right)\xi_b{}^C = 0. \quad (5.11)$$

The Einstein condition could also be weakened to conditions on the Cotton-York tensor, and this should impose additional genericity assumptions on the Weyl tensor to recover the version given in [TC12].

As a final remark, we may also ask ourselves whether the curvature condition in the above conjecture could possibly be weakened to say,  $C_{abcd}\xi^{aA}\xi^{bB}\xi^{cC'} = 0$ . This is after all yet another way of generalising the Petrov-Penrose type  $\{2, 1, 1\}$  condition from four to higher dimensions.

## A Spinor calculus in four and six dimensions

In low dimensions, the isomorphisms of Lie algebras  $\mathfrak{so}(4, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$  and  $\mathfrak{so}(6, \mathbb{C}) \cong \mathfrak{sl}(4, \mathbb{C})$  enable one to eliminate the use of  $\gamma$ -matrices, and one can develop a spinor calculus entirely based on irreducible representations of the special linear groups  $\mathrm{SL}(2, \mathbb{C})$  and  $\mathrm{SL}(4, \mathbb{C})$ . When working on a real vector space equipped with a split signature metric tensor, the corresponding isomorphisms are given by  $\mathfrak{so}(2, 2) \cong \mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$  and  $\mathfrak{so}(3, 3) \cong \mathfrak{sl}(4, \mathbb{R})$ , and the spinor calculus may be done over  $\mathbb{R}$ . For definiteness, we shall be working over  $\mathbb{C}$ .

### A.1 Four dimensions

Let  $(\mathfrak{V}, g_{ab})$  be a four-dimensional complex vector space equipped with a non-degenerate symmetric metric tensor. The double cover of the special orthogonal groups  $\mathrm{SO}(4, \mathbb{C})$  is isomorphic to  $\mathrm{SL}(2, \mathbb{C})^+ \times \mathrm{SL}(2, \mathbb{C})^-$ , where  $\mathrm{SL}(2, \mathbb{C})^{\pm}$  are two distinct copies of  $\mathrm{SL}(2, \mathbb{C})$ , whose standard representations are the two-dimensional chiral spinor spaces  $\mathfrak{S}^{\pm}$ . The  $\mathrm{SL}(2, \mathbb{C})$ -invariant bilinear forms on  $\mathfrak{S}^+$  and  $\mathfrak{S}^-$  are volume forms  $\varepsilon_{A'B'}$  and  $\varepsilon_{AB}$  respectively, so that we have the canonical identification  $(\mathfrak{S}^{\pm})^* \cong \mathfrak{S}^{\pm}$ . Any irreducible representation of  $\mathrm{SL}(2, \mathbb{C})^+$  must be a totally symmetric spinor, since by the two-dimensionality of  $\mathfrak{S}^+$ , skew-symmetry is eliminated by  $\varepsilon_{A'B'}$ , e.g.  $\alpha_{A'[B'C']} = \frac{1}{2}\alpha_{A'D'}\varepsilon_{B'C'}$ . Similar consideration applies to  $\mathrm{SL}(2, \mathbb{C})^-$ . Thus any irreducible tensor can be realised as a totally symmetric tensor product of spinors of either or both of the chiralities. Tensorial indices can be converted into spinorial ones by means of the Van der Waerden  $\gamma$ -matrices  $\gamma^a_{AA'}$ , which satisfy the identity

$$g_{ab}\gamma^a_{AA'}\gamma^b_{BB'} = 2\varepsilon_{A'B'}\varepsilon_{AB}. \quad (\text{A.1})$$

Thus, for a vector  $V^a$ , we have  $V^{AB'} = \frac{1}{\sqrt{2}}\gamma_a^{AB'}V^a$ , so that  $\mathfrak{V} \cong \mathfrak{S}^+ \otimes \mathfrak{S}^-$ , and we can omit the Van der Waerden  $\gamma$ -matrices altogether.<sup>10</sup> A 2-form  $F_{ab}$  splits into a self-dual part and an anti-self-dual part represented by symmetric spinors  $\phi_{A'B'}$  and  $\phi_{AB}$  respectively, i.e.

$$F_{ab} = \phi_{A'B'}\varepsilon_{AB} + \phi_{AB}\varepsilon_{A'B'} \quad \in \quad \odot^2\mathfrak{S}^+ \oplus \odot^2\mathfrak{S}^- \cong \wedge_+^2\mathfrak{V} \oplus \wedge_-^2\mathfrak{V}.$$

Similarly, the tracefree Ricci tensor, the Weyl tensor and the Cotton-York tensor can be expressed as

$$\begin{aligned}\Phi_{ab} &= \Phi_{ABA'B'} \\ C_{abcd} &= \Psi_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD} + \Psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'}, \\ A_{abc} &= A_{AA'B'C'}\varepsilon_{BC} + A_{A'ABC}\varepsilon_{B'C'},\end{aligned}$$

respectively, where  $\Psi_{A'B'C'D'}$  and  $\Psi_{ABCD}$  are the self-dual and anti-self-dual parts of  $C_{abcd}$  respectively, and  $A_{AA'B'C'}$  and  $A_{A'ABC}$  are the self-dual and anti-self-dual parts of  $A_{abc}$  respectively.

**Principal spinors** The identity (A.1) tells us that in four dimensions all chiral spinors are pure. Now, let us single out a spinor  $\xi^{A'}$ . We shall briefly re-express sections 2 and 3 in the language of two-spinor calculus. We first make the definitions

$$\mathfrak{S}^{\frac{1}{2}} := \langle \xi^{A'} \rangle, \quad \mathfrak{S}^{-\frac{1}{2}} := \mathfrak{S}^+, \quad \mathfrak{S}^0 := \mathfrak{S}^-.$$

As another consequence of the two-dimensionality of  $\mathfrak{S}^+$ , we can equivalently characterise  $\mathfrak{S}^{\frac{1}{2}}$  as

$$\mathfrak{S}^{\frac{1}{2}} = \{ \alpha^{A'} \in \mathfrak{S}^+ : \xi^{A'}\alpha_{A'} = 0 \}. \quad (\text{A.2})$$

The offshot of the characterisation (A.2) is that the operation of  $\ell$ -fold contraction with  $\xi^{A'}$  on an irreducible (i.e. totally symmetric) spinor  $\phi_{A'_1 A'_2 \dots A'_k}$  of valence  $k$  characterises it as an element of a vector subspace  $\mathfrak{S}^{\frac{k-2\ell+2}{2}}$  of  $\mathfrak{S}^{-\frac{k}{2}} := \odot^k \mathfrak{S}^{-\frac{1}{2}}$ , i.e.

$$\mathfrak{S}^{\frac{k-2\ell+2}{2}} := \{ \phi_{A'_1 A'_2 \dots A'_k} \in \mathfrak{S}^{-\frac{k}{2}} : \phi_{A'_1 A'_2 \dots A'_\ell A'_{\ell+1} \dots A'_k} \xi^{A'_1} \xi^{A'_2} \dots \xi^{A'_\ell} = 0 \},$$

for  $\ell = 1, \dots, k$ , which yields the filtration

$$\mathfrak{S}^{\frac{k}{2}} \subset \mathfrak{S}^{\frac{k}{2}-1} \subset \mathfrak{S}^{\frac{k}{2}-2} \subset \dots \subset \mathfrak{S}^{-\frac{k}{2}+1} \subset \mathfrak{S}^{-\frac{k}{2}}.$$

When  $k = 2$ , we get a filtration on  ${}^+\mathfrak{g} := \mathfrak{sl}(2, \mathbb{C})^+$ . In particular, the stabiliser  $\mathfrak{p} = {}^+\mathfrak{g}^0$  of  $\xi^{A'}$  consists of symmetric spinors  $\phi_{A'B'}$  satisfying  $\xi^{A'}\xi^{B'}\phi_{A'B'} = 0$ . We can fix a spinor  $\eta_{A'}$  such that  $\xi^{A'}\eta_{A'} = 1$  which gives a splitting  $\mathfrak{S}^+ = \mathfrak{S}^{\frac{1}{2}} \oplus \mathfrak{S}_{-\frac{1}{2}}$ . The grading element in  ${}^+\mathfrak{g}$  is then given by  $\xi_{(A'}\eta_{B')}$ . It is now clear that for any integer  $k$ , each summand  $\mathfrak{S}^{\frac{k-2\ell+2}{2}}/\mathfrak{S}^{\frac{k-2\ell+4}{2}}$  in the associated graded vector space is a one-dimensional irreducible  $\mathfrak{p}$ -module isomorphic to a  $\mathbb{C}$ -module  $\mathfrak{S}^{\frac{k-2\ell+2}{2}}$ , on which the grading element has eigenvalue  $\frac{k-2\ell+2}{2}$ . Of particular interest is the case  $k = 4$  where we obtain the  $\mathfrak{p}$ -invariant filtration

$${}^+\mathfrak{C}^2 \subset {}^+\mathfrak{C}^1 \subset {}^+\mathfrak{C}^0 \subset {}^+\mathfrak{C}^{-1} \subset {}^+\mathfrak{C}^{-2},$$

on the space  ${}^+\mathfrak{C}$  of self-dual Weyl tensors. The relation to the Petrov-Penrose types has already been given in the introduction.

To describe irreducible  $\mathfrak{p}$ -modules of mixed types, i.e. elements of  $(\odot^k \mathfrak{S}^+) \otimes (\odot^\ell \mathfrak{S}^-)$  for non-negative  $k$  and  $\ell$ , it suffices to tensor the  $\mathfrak{p}$ -invariant filtration on  $\odot^k \mathfrak{S}^+$  with  $\odot^\ell \mathfrak{S}^-$ . Thus, in the case of a vector  $V^{AB'}$ , we only have the two non-trivial algebraic conditions  $V_{AB'}\xi^{B'} = 0$  and  $V_{AB'} = 0$ . The former tells us that  $V^{AB'}$  takes the form  $V^{AB'} = \alpha^A \xi^{B'}$  for some spinor  $\alpha^A$ , and must be null.

<sup>10</sup>The normalisation factor  $\frac{1}{\sqrt{2}}$  has been added for convenience.

The same argument applies to the Cotton-York tensor, but one must be careful to distinguish between the self-dual part  $A_{AA'B'C'}$  and the anti-self-dual part  $A_{A'ABC}$ . These will fit into the two distinct filtrations

$$+\mathfrak{A}^{\frac{3}{2}} \subset +\mathfrak{A}^{\frac{1}{2}} \subset +\mathfrak{A}^{-\frac{1}{2}} \subset +\mathfrak{A}^{-\frac{3}{2}} = +\mathfrak{A}, \quad -\mathfrak{A}^{\frac{1}{2}} \subset -\mathfrak{A}^{-\frac{1}{2}} = -\mathfrak{A},$$

where  $+\mathfrak{A}$  and  $-\mathfrak{A}$  are the spaces of self-dual and anti-self-dual Cotton-York tensors respectively.

**Remark A.1** It is interesting to note how the maps  $\mathfrak{e}\Pi_i^j$  defined in (3.5) translate into spinor calculus. It is straightforward to check using that the  $\mathfrak{e}\Pi_{\pm 1}^1(C) = \mathfrak{e}\Pi_0^1(C) = \mathfrak{e}\Pi_0^2(C) = 0$  trivially in four dimensions, and there remains, up to constant factors,

$$\begin{aligned} \mathfrak{e}\Pi_{-2}^0(C) &= \xi^{A'} \xi^{B'} \xi^{C'} \xi^{D'} \Psi_{A'B'C'D'} \varepsilon^{AB} \varepsilon^{CD}, & \mathfrak{e}\Pi_{-1}^0(C) &= \xi^{A'} \xi^{B'} \xi^{C'} \Psi_{A'B'C'}^{D'} \varepsilon^{AB}, \\ \mathfrak{e}\Pi_0^0(C) &= \xi^{A'} \xi^{B'} \Psi_{A'B'}^{C'D'}, & \mathfrak{e}\Pi_1^0(C) &= \xi^{A'} \Psi_{A'}^{B'C'D'} \varepsilon^{AB} \varepsilon^{CD}, \\ \mathfrak{e}\Pi_0^3(C) &= \xi^{A'} \xi^{B'} \Psi^{ABCD}. \end{aligned}$$

In particular, we see that the map  $\mathfrak{e}\Pi_0^3$  projects into the anti-self-dual part of the Weyl tensor. See [FFS94] for a parallel with the Hermitian case. Similar considerations apply to  $\mathfrak{g}$  and  $\mathfrak{A}$ .

**Foliating spinors and recurrent spinors** The foliating spinor equation (5.16) and recurrent spinor equation (5.7) now read as

$$\xi^{A'} \xi^{B'} \nabla_{AA'} \xi_{B'} = 0, \quad \xi^{B'} \nabla_{AA'} \xi_{B'} = 0,$$

respectively, where  $\nabla_{AB'}$  stands for the Levi-Civita connection  $\nabla_a$ . Unlike in higher dimensions, there is no further splitting of these equations into irreducible parts and only the filtration (4.1) is relevant (see Remark 4.3). It is perhaps noteworthy that the four-dimensional version of equation (5.11) can be rewritten as

$$\xi^{B'} \nabla_{AA'} \xi_{B'} = \xi_{A'} \xi^{B'} \nabla_{AB'} f.$$

We refer to the literature, notably [PR84, PR86] for a detailed study of these spinorial equations and others.

## A.2 Six dimensions

Now, let  $(\mathfrak{V}, g_{ab})$  be a six-dimensional complex vector space equipped with a non-degenerate symmetric metric tensor. The chiral spinor spaces are dual to each other, i.e.  $(\mathfrak{S}^\pm)^* \cong \mathfrak{S}^\mp$ , and can be identified with the four-dimensional standard and dual representations of the spin group  $\mathrm{SL}(4, \mathbb{C})$ , the double cover of  $\mathrm{SO}(6, \mathbb{C})$ . One can then eliminate the use of primed indices in favour of the unprimed ones, so we shall write  $\mathfrak{S}$  for  $\mathfrak{S}^-$  and  $\mathfrak{S}^*$  for  $\mathfrak{S}^+$ . We can also convert tensor indices into a skew-symmetrised pair of indices by means of the skew-symmetric Van der Waerden  $\gamma$ -matrices<sup>11</sup>  $\frac{1}{2}\gamma^a_{AB}$  and  $\frac{1}{2}\gamma^{aAB}$ , which satisfy the identity

$$g_{ab} \gamma^a_{AB} \gamma^b_{CD} = 2\varepsilon_{ABCD}, \quad g_{ab} \gamma^{aAB} \gamma^{bCD} = 2\varepsilon^{ABCD}, \quad g_{ab} \gamma^a_{AB} \gamma^{bCD} = 4\delta_{[A}^C \delta_{B]}^D, \quad (\text{A.3})$$

where  $\varepsilon_{ABCD} = \varepsilon_{[ABCD]}$  and  $\varepsilon^{ABCD} = \varepsilon^{[ABCD]}$  are volume forms on  $\mathfrak{S}$  and  $\mathfrak{S}^*$  respectively satisfying the normalisation

$$\varepsilon_{ABCD} \varepsilon^{EFGH} = 24 \delta_{[A}^E \delta_B^F \delta_C^G \delta_{D]}^H.$$

Skew-symmetrised pairs of spinor indices can be raised and lowered by means of  $\frac{1}{2}\varepsilon_{ABCD}$  and  $\frac{1}{2}\varepsilon^{ABCD}$ , e.g.  $V_{AB} = \frac{1}{2}\varepsilon_{ABCD} V^{CD}$ . The isomorphism  $\wedge^2 \mathfrak{S} \cong \wedge^2 \mathfrak{S}^*$  is the spinorial counterpart of the metric isomorphism

<sup>11</sup>Again, the factor of  $\frac{1}{2}$  has been added for convenience.

$\mathfrak{V} \cong \mathfrak{V}^*$ . To summarise, we have the following identification between  $\bigwedge^\bullet \mathfrak{V}$  and tensor products of  $\mathfrak{S}$  and  $\mathfrak{S}^*$

$$\mathfrak{V} \cong \wedge^2 \mathfrak{S}, \quad \wedge^2 \mathfrak{V} \cong \mathfrak{S} \otimes \mathfrak{S}^*, \quad \wedge_+^3 \mathfrak{V} \cong \odot^2 \mathfrak{S}^*, \quad \wedge^3 \mathfrak{V} \cong \odot^2 \mathfrak{S}.$$

We note that by the identity (A.3), all spinors are pure in six dimensions. In addition to the above isomorphisms, we note that the tracefree Ricci tensor, the Weyl tensor, and the Cotton-York take the spinorial forms

$$\Phi_{ab} = \Phi_{ABCD}, \quad C_a{}^b{}_c{}^d = 8\delta_{[A}^C C_{B][E}^D] \delta_{F]}^H, \quad A_{ab}{}^c = 4A_{AB[C}{}^{[E} \delta_{D]}^F],$$

where  $\Phi_{ABCD} = \Phi_{[AB][CD]}$  satisfies  $\Phi_{[ABC]D} = 0$ ,  $C_{AB}^{CD} = C_{(AB)}^{(CD)}$  is tracefree, and  $A_{ABC}{}^D = A_{[AB]C}{}^D$  satisfies  $A_{[ABC]}{}^D = 0$  and  $A_{ABC}{}^A = 0$ .

**Principal spinors** Given a spinor  $\xi_A$ , there are now two distinct spinorial operations available: contraction with  $\xi_A$ , and skew-symmetrisation with  $\xi_A$ . These will give a six-dimensional analogue of the concept of a principal null spinor  $\xi_A$ . Following the notation of section 2, we have

$$\begin{aligned} \mathfrak{S}^{\frac{3}{4}} &= \langle \xi_A \rangle, & \mathfrak{S}^{-\frac{1}{4}} &= \mathfrak{S}^*, \\ \mathfrak{S}^{\frac{1}{4}} &= \{ \beta^A \in \mathfrak{S} : \beta^A \xi_A = 0 \}, & \mathfrak{S}^{-\frac{3}{4}} &= \mathfrak{S}. \end{aligned}$$

In fact, the operation of skew-symmetrisation appears in  $\mathfrak{S}^{\frac{3}{4}} = \{ \alpha_A \in \mathfrak{S}^* : \xi_{[A} \alpha_{B]} = 0 \}$ . We can extend this argument to spinors of any valence. Thus, again referring to the notation of section 2, a vector in  $\mathfrak{V}^{\frac{1}{2}}$  must satisfy  $\xi_{[A} V_{B]C} = 0$  or equivalently  $\xi_A V^{AB} = 0$ , in which case it takes the form  $V_{AB} = \xi_{[A} \alpha_{B]}$  for some spinor  $\alpha_A$ .

A 2-form, or equivalently, an element of the Lie algebra  $\mathfrak{so}(6, \mathbb{C})$ , can be represented by a tracefree spinor  $\phi_A{}^B$ . The maps (2.22) defining the subalgebras  $\mathfrak{g}^i$  determined by  $\xi_A$  can then simply be expressed as

$${}^9\Pi_1(\phi) := \xi_{[A} \phi_{B]}{}^C \xi_C, \quad {}^9\Pi_0^0(\phi) := \phi_A{}^B \xi_B, \quad {}^9\Pi_0^1(\phi) := \xi_{[A} \phi_{B]}{}^C - \frac{1}{3} \delta_{[A}^C \phi_{B]}{}^D \xi_D.$$

The classification of curvature tensors is virtually identical as in the general higher-dimensional case. Here, we record the spinorial version of the maps (3.5):

$$\begin{aligned} {}^e\Pi_{-2}^0(C) &:= \xi_{[A} C_{B][C}^{EF} \xi_D] \xi_E \xi_F, \\ {}^e\Pi_{-1}^0(C) &:= \xi_{[A} C_{B]C}^{DE} \xi_D \xi_E, \\ {}^e\Pi_{-1}^1(C) &:= \xi_{[A} C_{B][C}^{EF} \xi_D] \xi_F - \frac{1}{4} \delta_{[A}^E C_{B][C}^{FG} \xi_D] \xi_F \xi_G - \frac{1}{4} \delta_{[C}^E C_{D][A}^{FG} \xi_B] \xi_F \xi_G, \\ {}^e\Pi_0^0(C) &:= C_{AB}^{CD} \xi_C \xi_D, \\ {}^e\Pi_0^1(C) &:= \xi_{[A} C_{B]C}^{DE} \xi_E - \frac{1}{3} \delta_{[A}^D C_{B]C}^{EF} \xi_E \xi_F, \\ {}^e\Pi_0^3(C) &:= \xi_{[A} C_{B][C}^{EF} \xi_D] - \frac{2}{5} \delta_{[A}^{(E} C_{B][C}^{F)G} \xi_D] \xi_G - \frac{2}{5} \delta_{[C}^{(E} C_{D][A}^{F)G} \xi_B] \xi_G + \frac{1}{10} \delta_{[A}^{(E} C_{B][C}^{GH} \delta_{D]}^{F)} \xi_G \xi_H, \\ {}^e\Pi_1^0(C) &:= C_{AB}^{CD} \xi_D, \\ {}^e\Pi_1^1(C) &:= \xi_{[A} C_{B]C}^{DE} - \frac{1}{2} \delta_{[A}^{(D} C_{B]C}^{E)F} \xi_F. \end{aligned}$$

**Remark A.2** The classification of  $C_{AB}^{CD}$  according to its algebraic relation with one or more copies of a spinor  $\xi_A$  was first put forward by Jeffries [Jef95]. However, his classification has the drawback that it is not irreducible in the sense that the maps he defines are not saturated with symmetries. This slight oversight is rectified in the above expressions.

Similarly, the maps (3.2) defining the classes of Cotton-York tensors have spinorial forms

$$\begin{aligned}
\mathfrak{A}\Pi_{-\frac{3}{2}}^0(A) &:= \xi_{[A} A_{B C][D}{}^F \xi_E] \xi_F \\
\mathfrak{A}\Pi_{-\frac{1}{2}}^0(A) &:= \xi_{[A} A_{B C]D}{}^E \xi_E, \\
\mathfrak{A}\Pi_{-\frac{1}{2}}^1(A) &:= A_{AB[C}{}^E \xi_{D]} \xi_E + A_{CD[A}{}^E \xi_{B]} \xi_E, \\
\mathfrak{A}\Pi_{-\frac{1}{2}}^2(A) &:= \xi_{[A} A_{B C][D}{}^F \xi_E] - \frac{1}{4} \delta_{[A}^F A_{B C][D}{}^G \xi_E] \xi_G - \frac{1}{4} \xi_{[A} A_{B C][D}{}^G \delta_E^F \xi_G, \\
\mathfrak{A}\Pi_{\frac{1}{2}}^0(A) &:= A_{ABC}{}^D \xi_D, \\
\mathfrak{A}\Pi_{\frac{1}{2}}^1(A) &:= \xi_{[A} A_{B C]D}{}^E - \frac{1}{2} \delta_{[A}^E A_{B C]D}{}^F \xi_F, \\
\mathfrak{A}\Pi_{\frac{1}{2}}^2(A) &:= A_{AB[C}{}^E \xi_{D]} + A_{CD[A}{}^E \xi_{B]} - \frac{2}{5} \left( A_{AB[C}{}^F \delta_{D]}^E \xi_F + A_{CD[A}{}^F \delta_{B]}^E \xi_F \right).
\end{aligned}$$

**Foliating spinors and recurrent spinors** The foliating spinor equation (5.16) and the recurrent spinor equation (5.7) can be expressed as

$$(\xi_D \nabla^{DA} \xi_{[B}) \xi_{C]} = 0, \quad (\text{A.4})$$

$$(\nabla^{AB} \xi_{[C}) \xi_{D]} = 0, \quad (\text{A.5})$$

respectively, where  $\nabla_{AB}$  stands for the Levi-Civita connection  $\nabla_a$ . Equation (A.4) splits into two irreducible parts

$$(\xi_D \nabla^{DA} \xi_{[B}) \xi_{C]} - \frac{1}{3} (\xi_D \nabla^{DE} \xi_E) \delta_{[B}^A \xi_{C]} = 0, \quad (\text{A.6})$$

$$\xi_A \nabla^{AB} \xi_B = 0, \quad (\text{A.7})$$

which are equivalent to (5.11) and (5.12) respectively. On the other hand, equation (A.5) splits into two irreducible parts

$$(\nabla^{AB} \xi_{[C}) \xi_{D]} - \left( \nabla^{[A|E} \xi_E \right) \delta_{[C}^{B]} \xi_{D]} - \left( \xi_E \nabla^{[A|E} \xi_{[C} \right) \delta_{D]}^{B]} - \frac{1}{3} (\xi_E \nabla^{EF} \xi_F) \delta_{[A}^C \delta_{B]}^D = 0, \quad (\text{A.8})$$

$$\xi_A \nabla^{BC} \xi_C + \xi_C \nabla^{CB} \xi_A = 0, \quad (\text{A.9})$$

which are equivalent to (5.15) and (5.14) respectively.

Finally, a (positive) twistor-spinor satisfies

$$\nabla^{AB} \xi_C + \frac{2}{3} \delta_C^{[A} \nabla^{B]E} \xi_E = 0.$$

A little algebra yields

$$\begin{aligned}
& \left( \nabla^{AB} \xi_{[C} \right) \xi_{D]} - \frac{2}{3} \xi_{[C} \delta_{D]}^{[A} \nabla^{B]E} \xi_E = 0, \\
& \xi_E \left( \nabla^{E[A} \xi_{[C} \right) \delta_{D]}^{B]} - \frac{1}{3} \xi_{[C} \delta_{D]}^{[A} \left( \nabla^{B]E} \xi_E \right) - \frac{1}{3} \delta_{[C}^A \delta_{D]}^B (\xi_E \nabla^{EF} \xi_F) = 0,
\end{aligned}$$

from which we deduce that  $\xi_A$  satisfies equations (A.8) and (A.6). This proves Proposition 5.26.

## B Conformal structures

In this appendix, we collect a few facts and conventions pertaining to conformal geometry. We roughly follow [BEG94], although our staggering of indices differs from theirs.

A *conformal structure* on a smooth or complex manifold  $\mathcal{M}$  is an equivalence class of metrics  $[g_{ab}]$  on  $\mathcal{M}$ , whereby two metrics  $\hat{g}_{ab}$  and  $g_{ab}$  belong to the same class if and only if

$$\hat{g}_{ab} = \Omega^2 g_{ab}, \quad (\text{B.1})$$

for some non-vanishing smooth or homomorphic function  $\Omega$  on  $\mathcal{M}$ . The respective Levi-Civita connections  $\nabla_a$  and  $\hat{\nabla}_a$  of  $g_{ab}$  and  $\hat{g}_{ab}$  are then related by

$$\hat{\nabla}_a V^b = \nabla_a V^b + Q_{ac}{}^b V^c, \quad Q_{abc} := Q_{ab}{}^d g_{dc} = 2\Upsilon_{(a} g_{b)c} - \Upsilon_c g_{ab},$$

where  $\Upsilon_a := \Omega^{-1} \nabla_a \Omega$ .

**Spinor bundles** We first note that under a rescaling (B.1), the Van der Waerden  $\gamma$ -matrices can be chosen to transform as

$$\gamma_{aA}{}^{B'} \mapsto \hat{\gamma}_{aA}{}^{B'} = \Omega \gamma_{aA}{}^{B'}, \quad \gamma_{aB'}{}^A \mapsto \hat{\gamma}_{aB'}{}^A = \Omega \gamma_{aB'}{}^A,$$

where  $\hat{\gamma}_{aA}{}^{B'}$  and  $\hat{\gamma}_{aB'}{}^A$  denote the Van der Waerden  $\gamma$ -matrices for the metric  $\hat{g}_{ab}$ . In addition, we can choose the  $\text{Spin}(2m, \mathbb{C})$ -invariant bilinear forms on  $\mathcal{S}$  to rescale with a conformal weight of 1, and their dual with a conformal weight of  $-1$ . For instance,  $\varepsilon_{A'B'} \mapsto \hat{\varepsilon}_{A'B'} = \Omega \varepsilon_{A'B'}$  when  $m$  is even,  $\varepsilon^{A'B} \mapsto \hat{\varepsilon}^{A'B} = \Omega^{-1} \varepsilon^{A'B}$  when  $m$  is odd, and so on. This means in particular that the quantities  $\gamma_a{}^{AB'}$  and  $\gamma^a{}_{AB'}$  when  $m$  is even, and  $\gamma_a{}^{A'B'}$  and  $\gamma^a{}_{A'B'}$ , and their unprimed counterparts, when  $m$  is odd, have conformal weight 0. Then the spin connection  $\hat{\nabla}_a$  is related to  $\nabla_a$  by

$$\begin{aligned} \hat{\nabla}_a \xi^{B'} &= \nabla_a \xi^{B'} - \frac{1}{2} \Upsilon_b \gamma^b{}_{aC'}{}^{B'} \xi^{C'} + \frac{1}{2} \Upsilon_a \xi^{B'} \\ &= \nabla_a \xi^{B'} - \frac{1}{2} \Upsilon_b \gamma^b{}_{C'}{}^D \gamma_{aD}{}^{B'} \xi^{C'}, \end{aligned} \quad (\text{B.2})$$

and similarly for unprimed and dual spinors. This connection can be seen to preserve the hatted  $\gamma$ -matrices *and* the hatted bilinear forms on  $\mathcal{S}$ . This agrees with the convention of [PR84] but differs from the more standard convention, used in [LM89] for instance.

**Curvature** In conformal geometry, it is more convenient to use the alternative decomposition to (5.3)

$$R_{abcd} = C_{abcd} - 4g_{[c|[a} P_{b]d]}, \quad P_{ab} := \frac{1}{2-n} \Phi_{ab} - R \frac{1}{2n(n-1)} g_{ab}. \quad (\text{B.3})$$

where  $P_{ab}$  is the *Schouten* or *Rho tensor*. This tensor and its trace  $P$  transform as

$$\hat{P}_{ab} = P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{2} \Upsilon_c \Upsilon^c g_{ab}, \quad \hat{P} = \Omega^{-2} \left( P - \nabla^c \Upsilon_c - \frac{n-2}{2} \Upsilon^c \Upsilon_c \right). \quad (\text{B.4})$$

Finally, the Cotton-York tensor

$$A_{abc} := 2\nabla_{[b} P_{c]a} = -(n-3)\nabla^d C_{dabc},$$

where the expression on the RHS follows from the contracted Bianchi identity, transforms as

$$\hat{A}_{abc} = A_{abc} - \Upsilon^d C_{dabc}.$$

## C Representation theory

In this appendix, we describe the Lie algebra  $\mathfrak{so}(2m, \mathbb{C})$  and the Lie parabolic subalgebra  $\mathfrak{p}$  preserving a pure spinor in the language of representation theory given in [FH91, BE89, ČS09]. We also collect the descriptions of the various irreducible representations occurring in Propositions 3.5, 3.3, and 4.2 in the classification of curvature tensors and intrinsic torsion.

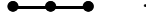
### C.1 Case $m > 2$

### C.1.1 The Lie algebras $\mathfrak{so}(2m, \mathbb{C})$

We shall represent the complex simple Lie algebra  $\mathfrak{g} := \mathfrak{so}(2m, \mathbb{C})$  in terms of its Dynkin diagram

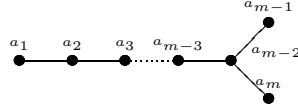


when  $m \geq 3$ . When  $m = 3$ , it is more usual to rotate the Dynkin diagram clockwise and flatten it, and use the Dynkin diagram for  $\mathfrak{sl}(4, \mathbb{C}) \cong \mathfrak{so}(6, \mathbb{C})$ , i.e.





Here, a node corresponds to a simple root in the dual Cartan subalgebra  $\mathfrak{h}^*$  of  $\mathfrak{so}(2m, \mathbb{C})$ , and an edge between two nodes is related to the pairing between the corresponding roots with respect to the Killing form on  $\mathfrak{so}(2m, \mathbb{C})$ .

**Irreducible representations of  $\mathfrak{so}(2m, \mathbb{C})$**  Recall that there is a one-to-one correspondence between finite irreducible representations of  $\mathfrak{g}$  and dominant weights for  $\mathfrak{g}$  in  $\mathfrak{h}^*$ . These can be described by means of the following notation



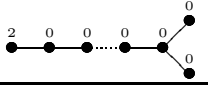
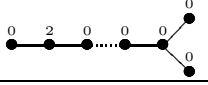
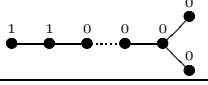
where, for all  $i$ , the  $a_i$  are the coefficients of the dominant weight with respect to a weight basis dual to the simple coroots of the Lie algebra. These coefficients are always non-negative integers. We shall thus adopt this dominant weight notation to denote irreducible representation of  $\mathfrak{g}$ . The dimension of such an irreducible representation can be computed by Weyl's character formulae. For the standard and adjoint representations, we have

g-mod	Dominant weight	Dimension
$\mathfrak{V}$		$m$
$\mathfrak{g}$		$m(2m-1)$

with the proviso that when  $m = 3$ , the dominant weight of  $\mathfrak{g}$  is  $\overset{1}{\bullet} - \overset{0}{\bullet} \overset{1}{\bullet}$ .

**The Cartan product** Recall that the two irreducible finite representations,  $\mathfrak{P}$  and  $\mathfrak{R}$  say, of  $\mathfrak{g}$ , the tensor product representation  $\mathfrak{P} \otimes \mathfrak{R}$  of  $\mathfrak{g}$  splits as a direct sum of irreducible representations of  $\mathfrak{g}$ . The *Cartan product* of  $\mathfrak{P}$  and  $\mathfrak{R}$ , denoted  $\mathfrak{P} \odot \mathfrak{R}$ , is the unique irreducible representation occurring in  $\mathfrak{P} \otimes \mathfrak{R}$  with multiplicity one. In the case  $\mathfrak{g} = \mathfrak{so}(2m, \mathbb{C})$ , the Cartan product coincides with the tracefree symmetric tensor product  $\odot_{\circ}$ . In fact, the irreducible tensor representations describing the tracefree Ricci tensor, the Weyl tensor, and the Cotton-York tensor are simply the Cartan products  $\mathfrak{F} \cong \mathfrak{V} \odot \mathfrak{V}$ ,  $\mathfrak{C} \cong \mathfrak{g} \odot \mathfrak{g}$ , and  $\mathfrak{A} \cong \mathfrak{V} \odot \mathfrak{g}$ . Conveniently, the dominant weight of the Cartan product  $\mathfrak{P} \odot \mathfrak{R}$  is equal to the sum of the dominant weights of  $\mathfrak{P}$  and  $\mathfrak{R}$ . Further properties of the Cartan product are given in [Eas04]. From this, we

derive the following table

$\mathfrak{g}$ -mod	Dominant weight	Dimension
$\mathfrak{F}$		$(2m-1)(m+1)$
$\mathfrak{C}$		$\frac{1}{3}m(m+1)(2m+1)(2m-3)$
$\mathfrak{A}$		$\frac{8}{3}m(m+1)(m-1)$

with the proviso that when  $m = 3$ , the dominant weights of  $\mathfrak{C}$  and of  $\mathfrak{A}$  are  and  respectively.

### C.1.2 The parabolic Lie subalgebra $\mathfrak{p}$

A (standard) parabolic Lie subalgebra  $\mathfrak{p}$  of a complex simple Lie algebra  $\mathfrak{g}$  can also be represented by a Dynkin diagram, albeit a ‘mutilated’ one. To understand the nature of the mutilation, we return to our specific situation where  $\mathfrak{g} = \mathfrak{so}(2m, \mathbb{C})$  and  $\mathfrak{p}$  stabilises a projective positive pure spinor, or equivalently an  $\alpha$ -plane in  $\mathfrak{V}$ . Our starting point will be the Dynkin diagram for  $\mathfrak{g}$ , but we shall cross the node corresponding to the positive spinor representation, i.e.



for  $m \geq 3$  and  $m = 3$  respectively. For  $\beta$ -planes, one crosses out the other spinor node.<sup>12</sup>

We can understand this notation further in the light of the Levy decomposition

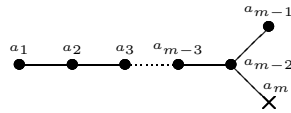
$$\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where  $\mathfrak{g}_0 \cong \mathfrak{gl}(m, \mathbb{C})$  is reductive, and  $\mathfrak{g}_1$  is nilpotent. It becomes clear that the simple part  $\mathfrak{sl}_0 \cong \mathfrak{sl}(m, \mathbb{C})$  of  $\mathfrak{g}_0$  can be recovered by deleting the crossed nodes of the mutilated Dynkin diagram for  $\mathfrak{p}$ , i.e.

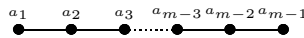


for  $m \geq 3$  and  $m = 3$  respectively.

**Irreducible representations of  $\mathfrak{so}(2m, \mathbb{C})$**  Analogously to the theory of simple Lie algebras, there is a one-to-one correspondence between finite irreducible representations of  $\mathfrak{p}$  and dominant integral weights for  $\mathfrak{p}$  in  $(\mathfrak{h} \cap \mathfrak{sl}_0)^*$ , and these are described in the Dynkin diagram notation by



where  $a_i$  are integers<sup>13</sup> for all  $i$ , and non-negative for  $i = 1, \dots, m-1$ . Here, dominant integral for  $\mathfrak{p}$  means dominant integral for  $\mathfrak{sl}_0$ , the simple part of  $\mathfrak{p}$ . Thus, deleting the crossed node of the mutilated Dynkin diagram yields the  $\mathfrak{sl}_0$ -dominant weight



<sup>12</sup>Which node is crossed out is usually a matter of convention, and the opposite choice can be found in the literature.

<sup>13</sup>The coefficient  $a_m$  does not need to be integral at the Lie algebra level, but does have to at the Lie group level.



whose dimension can be easily computed by means of a Weyl character formula.

To recover the full weight for  $\mathfrak{p}$ , one needs to account for the coefficient  $a_m$  over the crossed node. Its value is in fact related to the eigenvalue of the grading element  $E$  associated to the parabolic Lie subalgebra on the highest weight vector of the irreducible  $\mathfrak{p}$ -module under consideration. There is a straightforward algorithm involving the inverse Cartan matrix for  $\mathfrak{g}$  to determine this value: if  $i$  is the eigenvalue of the grading element  $E$  on the highest weight vector, then

$$i = \frac{1}{4} (2a_1 + 4a_2 + 6a_3 + \dots + 2(m-2)a_{m-2} + (m-2)a_{m-1} + ma_m) ,$$

as explained in [ČS09].

To make contact with the analysis of the main text, we identify the standard representation  $\mathfrak{V}_{\frac{1}{2}}$  of  $\mathfrak{g}_0$  with the spinor module  $\mathfrak{S}_{-\frac{m-2}{4}}$  by means of (2.6). An element of the irreducible  $\mathfrak{g}_0$ -module

$$\begin{array}{c} 0 \quad 0 \quad \dots \quad 0 \quad 1 \quad 0 \quad \dots \quad 0 \quad 0 \\ \bullet \quad \bullet \quad \dots \quad \bullet \quad \uparrow \quad \bullet \quad \dots \quad \bullet \quad \bullet \\ \text{\scriptsize } i^{\text{th}} \text{ node} \end{array} \cong \wedge^i \mathfrak{V}_{\frac{1}{2}} ,$$

is then a spinor  $\phi_{A_1 \dots A_k} = \phi_{[A_1 \dots A_k]}$  in  $\wedge^k \mathfrak{S}_{-\frac{m-2}{4}}$ , or dually as  $\phi^{A_1 \dots A_{m-1-k}} = \phi^{[A_1 \dots A_{m-1-k}]}$  in  $\wedge^k \mathfrak{S}_{-\frac{m-2}{4}}$ . It is then straightforward to express any other irreducible representations in terms of these ‘building blocks’ by using the Cartan product. For instance, an element in  $\begin{array}{c} 1 \quad 1 \quad 0 \quad \dots \quad 0 \\ \bullet \quad \bullet \quad \bullet \quad \dots \quad \bullet \end{array}$  takes the form  $\phi_{ABC} = \phi_{A[BC]}$  and satisfies  $\phi_{[ABC]} = 0$ . Of course, one also needs to know the value of the coefficient over the crossed node to determine how many  $\xi^{aA}$  and  $\eta^a_A$  should be hooked to our spinor, and how many copies of  $\xi^{A'}$ ,  $\eta_{A'}$  and  $\omega_{ab}$  are tensored with it.

In the following tables, we have collected the irreducible  $\mathfrak{p}$ -modules relevant to this article, their corresponding  $\mathfrak{g}_0$ -modules,<sup>14</sup> dominant weights and dimensions, with the proviso that when  $m = 3$

- the dominant weights of  $\mathfrak{g}_1^0$  and  $\mathfrak{W}_{-\frac{3}{2}}^0$  are given by  $\begin{array}{c} 1 \quad 0 \quad 1 \\ \times \quad \bullet \quad \bullet \end{array}$  and  $\begin{array}{c} -2 \quad 0 \quad 0 \\ \times \quad \bullet \quad \bullet \end{array}$ , respectively, and
- the  $\mathfrak{p}$ -module  $\mathfrak{C}_0^2$  does not occur.

We note that the  $\mathfrak{p}$ -irreducibles of  $\mathfrak{F}$ ,  $\mathfrak{C}$  and  $\mathfrak{A}$  are conveniently described in terms of Cartan products of  $\mathfrak{p}$ -irreducibles of  $\mathfrak{V}$  and  $\mathfrak{g}$ .

$\mathfrak{p}$ -mod	$\mathfrak{g}_0$ -mod	Dominant weight	$\mathfrak{p}$ -mod	$\mathfrak{g}_0$ -mod	Dominant weight	Dimension
$\mathfrak{V}_{\frac{1}{2}}$	$\mathfrak{V}_{\frac{1}{2}}$	$\begin{array}{c} 1 \quad 0 \quad 0 \quad \dots \quad 0 \quad 0 \\ \bullet \quad \bullet \quad \bullet \quad \dots \quad \bullet \quad \bullet \\ \text{\scriptsize } \times \end{array}$	$\mathfrak{V}_{-\frac{1}{2}} / \mathfrak{V}_{\frac{1}{2}}$	$\mathfrak{V}_{-\frac{1}{2}}$	$\begin{array}{c} 0 \quad 0 \quad 0 \quad \dots \quad 0 \quad 0 \\ \bullet \quad \bullet \quad \bullet \quad \dots \quad \bullet \quad \bullet \\ \text{\scriptsize } \times \end{array}$	$m$

Table 1: Irreducible  $\mathfrak{p}$ -modules of  $\mathfrak{V}$

## C.2 Case $m = 2$

### C.2.1 The Lie algebra of $\mathfrak{so}(4, \mathbb{C})$

In four dimensions,  $\mathfrak{so}(4, \mathbb{C})$  is not simple, but is isomorphic to  $\mathfrak{sl}(2, \mathbb{C})^+ \times \mathfrak{sl}(2, \mathbb{C})^-$ , so that the corresponding Dynkin diagram is

$$\bullet \quad \bullet$$

<sup>14</sup>We have abbreviated ‘ $\mathfrak{p}$ -module’ and ‘ $\mathfrak{g}_0$ -module’ to ‘ $\mathfrak{p}$ -mod’ and ‘ $\mathfrak{g}_0$ -mod’ respectively.



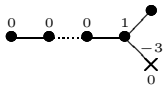
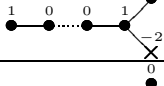
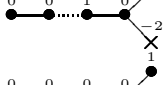
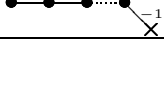
$\mathfrak{p}$ -mod	$\mathfrak{g}_0$ -mod	Dominant weight	Dimension
$\mathfrak{W}_{-\frac{3}{2}}^1$	$\mathfrak{V}_{-\frac{1}{2}} \odot \wedge^2 \mathfrak{V}_{-\frac{1}{2}}$		$\frac{1}{3}m(m^2 - 1)$
$\mathfrak{W}_{-\frac{1}{2}}^1$	$\mathfrak{V}_{\frac{1}{2}} \odot \wedge^2 \mathfrak{V}_{-\frac{1}{2}}$		$\frac{1}{2}m(m+1)(m-2)$
$\mathfrak{W}_{-\frac{3}{2}}^0$	$\wedge^3 \mathfrak{V}_{-\frac{1}{2}}$		$\frac{1}{3!}m(m-1)(m-2)$
$\mathfrak{W}_{-\frac{1}{2}}^0$	$\mathfrak{V}_{-\frac{1}{2}}$		$m$

Table 6: Irreducible  $\mathfrak{p}$ -modules of  $\mathfrak{W}$

particular, if the irreducible  $\mathfrak{p}$ -module is purely positive spinorial, i.e. with  $\ell = 0$ , then the resulting  $\mathfrak{p}$ -modules will be one-dimensional. For instance, each  $\mathfrak{p}$ -module  ${}^+\mathfrak{C}^j/{}^+\mathfrak{C}^{j+1}$  is given by  $\times \begin{smallmatrix} 2j \\ 0 \end{smallmatrix} \bullet$  with  $j = 0, \pm 1, \pm 2$ . The details are left to the reader.

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